HEIGHT ZETA FUNCTIONS OF EQUIVARIANT COMPACTIFICATIONS OF SEMI-DIRECT PRODUCTS OF ALGEBRAIC GROUPS

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Abstract. We apply the theory of height zeta functions to study the asymptotic distribution of rational points of bounded height on projective equivariant compactifications of semi-direct products.

Contents

Introduction

Let X be a smooth projective variety over a number field F and L a very ample line bundle on X. An adelic metrization $\mathcal{L} = (L, \|\cdot\|)$ on L induces a height function

$$\mathsf{H}_{\mathcal{L}}\colon X(F) \to \mathbb{R}_{>0},$$

let

$$\mathsf{N}(X^\circ, \mathcal{L}, \mathsf{B}) := \#\{x \in X^\circ(F) \,|\, \mathsf{H}_{\mathcal{L}}(x) \leq \mathsf{B}\}, \quad X^\circ \subset X,$$

be the associated counting function for a subvariety X°. Manin's program, initiated in [FMT89] and significantly developed over the last 10 years, relates the asymptotic of the counting function $\mathsf{N}(X^\circ, \mathcal{L}, \mathsf{B})$, as $\mathsf{B} \to \infty$, for a suitable Zariski open $X^\circ \subset X$, to global geometric invariants of the underlying variety X. By general principles of diophantine geometry, such a connection can be expected for varieties with sufficiently positive anticanonical line bundle $-K_X$, e.g., for Fano varieties. Manin's conjecture asserts that

$$\mathsf{N}(X^\circ, -\mathcal{K}_X, \mathsf{B}) = c \cdot \mathsf{B} \log(\mathsf{B})^{r-1}, \tag{0.1}$$

where r is the rank of the Picard group $\mathrm{Pic}(X)$ of X, at least over a finite extension of the ground field. The constant c admits a conceptual interpretation, its main ingredient is a Tamagawa-type number introduced by Peyre [Pey95].

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For recent surveys highlighting different aspects of this program, see, e.g., [Tsc09], [CL10], [Bro07], [Bro09].

Several approaches to this problem have evolved:

- passage to (universal) torsors combined with lattice point counts;
- variants of the circle method;
- ergodic theory and mixing;
- height zeta functions and spectral theory on adelic groups.

The universal torsor approach has been particularly successful in the treatment of del Pezzo surfaces, especially the singular ones. This method works best over $\mathbb{Q}$; applying it to surfaces over more general number fields often presents insurmountable difficulties, see, e.g., [dlBF04]. Here we will explain the basic principles of the method of height zeta functions of equivariant compactifications of linear algebraic groups and apply it to semi-direct products; this method is insensitive to the ground field. The spectral expansion of the height zeta function involves 1-dimensional as well as infinite-dimensional representations, see Section 3 for details on the spectral theory. We show that the main term appearing in the spectral analysis, namely, the term corresponding to 1-dimensional representations, matches precisely the predictions of Manin's conjecture, i.e., has the form (0.1). The analogous result for the universal torsor approach can be found in [Pey98] and for the circle method applied to universal torsors in [Pey01].

Furthermore, using the tools developed in Section 3, we provide new examples of rational surfaces satisfying Manin's conjecture.

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1. Geometry

In this section, we collect some general geometric facts concerning equivariant compactifications of solvable linear algebraic groups. Here we work over an algebraically closed field of characteristic 0.

Let G be a linear algebraic group. In dimension 1, the only examples are the additive group $\mathbb{G}_a$ and the multiplicative group $\mathbb{G}_m$. Let

$$\mathfrak{X}(G)^\times := \mathrm{Hom}(G, \mathbb{G}_m),$$

the group of algebraic characters of G. In the situations we consider, this is a torsion-free $\mathbb{Z}$-module of finite rank (see [Bor91, Section 8], for conditions insuring this property).

Let X be a projective equivariant compactification of G. After applying equivariant resolution of singularities, if necessary, we may assume that X is smooth and that the boundary

$$X \setminus G = D = \cup_\iota D_\iota,$$

is a divisor with normal crossings. Here D_ι are irreducible components of D. Let $\mathrm{Pic}^G(X)$ be the group of equivalence classes of G-linearized line bundles on X.

Generally, we will identify divisors, associated line bundles, and their classes in $\mathrm{Pic}(X)$, resp. $\mathrm{Pic}^G(X)$.

Proposition 1.1. *Let X be a smooth and proper equivariant compactification of a connected solvable linear algebraic group G. Then,*

(1) *we have an exact sequence*

$$0 \to \mathfrak{X}(G)^\times \to \mathrm{Pic}(X)^G \to \mathrm{Pic}(X) \to 0,$$

(2) $\mathrm{Pic}^G(X) = \oplus_{\iota\in\mathcal{I}}\mathbb{Z}D_\iota$, *and*

(3) *the closed cone of pseudo-effective divisors of X is spanned by the boundary components:*

$$\Lambda_{\mathrm{eff}}(X) = \sum_{\iota\in\mathcal{I}} \mathbb{R}_{\geq 0} D_\iota.$$

Proof. The first claim follows from the proof of [MFK94, Proposition 1.5]. The crucial point is to show that the Picard group of G is trivial. As an algebraic variety, a connected solvable group is a product of an algebraic torus and an affine space. The second assertion holds since every finite-dimensional representation of a solvable group has a fixed vector. For the last statement, see [HT99, Theorem 2.5]. □

Proposition 1.2. *Let X be a smooth and proper equivariant compactification for the left action of a linear algebraic group. Then the right invariant top degree differential form ω satisfies*

$$-\mathrm{div}(\omega) = \sum_{\iota\in\mathcal{I}} d_\iota D_\iota,$$

where $d_\iota > 0$. The same result holds for the right action and the left invariant form.

Proof. See [HT99, Theorem 2.7]. □

Proposition 1.3. *Let X be a smooth and proper equivariant compactification of a connected linear algebraic group. Let $f : X \to Y$ be a birational morphism to a normal projective variety Y. Then Y is an equivariant compactification of G such that the contraction map f is a G-morphism.*

Proof. Choose an embedding $Y \hookrightarrow \mathbb{P}^N$, and let L be the pull back of $\mathcal{O}(1)$ on X. Since Y is normal, Zariski's main theorem implies that the image of the complete linear series $|L|$ is isomorphic to Y. This linear series carries a G-linearization [MFK94, Corollary 1.6]. Now we apply the same argument as in the proof of [HT99, Corollary 2.4] to this linear series, and our assertion follows. □

The simplest solvable groups are $\mathbb{G}_a$ and $\mathbb{G}_m$, as well as their products. New examples arise as semi-direct products. For example, let

$$\begin{array}{rcl} \varphi_d : \mathbb{G}_m & \to & \mathbb{G}_m = \mathrm{GL}_1, \\ a & \mapsto & a^d \end{array}$$

and put

$$G_d := \mathbb{G}_a \rtimes_{\varphi_d} \mathbb{G}_m,$$

where the group law is given by

$$(x, a) \cdot (y, b) = (x + \varphi_d(a)y, ab).$$

It is easy to see that $G_d \simeq G_{-d}$.

One of the central themes in birational geometry is the problem of classification of algebraic varieties. The classification of G-varieties, i.e., varieties with G-actions, is already a formidable task. The theory of toric varieties, i.e., equivariant compactifications of $G = \mathbb{G}_m^n$, is very rich, and provides a testing ground for many conjectures

in algebraic and arithmetic geometry. See [HT99] for first steps towards a classification of equivariant compactifications of $G = \mathbb{G}_a^n$, as well as [Sha09], [AS09], [Arz10] for further results in this direction.

Much less is known concerning equivariant compactifications of other solvable groups; indeed, classifying equivariant compactifications of G_d is already an interesting open question. We now collect several results illustrating specific phenomena connected with noncommutativity of G_d and with the necessity to distinguish actions on the left, on the right, or on both sides. These play a role in the analysis of height zeta functions in following sections. First of all, we have

Lemma 1.4. *Let X be a biequivariant compactification of a semi-direct product $G \rtimes H$ of linear algebraic groups. Then X is a one-sided (left- or right-) equivariant compactification of $G \times H$.*

Proof. Fix one section $s : H \to G \rtimes H$. Define a left action by

$$(g, h) \cdot x = g \cdot x \cdot s(h)^{-1},$$

for any $g \in G$, $h \in H$, and $x \in X$. $\square$

In particular, there is no need to invoke noncommutative harmonic analysis in the treatment of height zeta functions of biequivariant compactifications of general solvable groups since such groups are semi-direct products of tori with unipotent groups and the lemma reduces the problem to a one-sided action of the *direct* product. Height zeta functions of direct products of additive groups and tori can be treated by combining the methods of [BT98] and [BT96a] with [CLT02], see Theorem 2.1. However, Manin's conjectures are still open for one-sided actions of unipotent groups, even for the Heisenberg group.

The next observation is that the projective plane $\mathbb{P}^2$ is an equivariant compactification of G_d, for any d. Indeed, the embedding

$$(x, a) \mapsto (a : x : 1) \in \mathbb{P}^2$$

defines a left-sided equivariant compactification, with boundary a union of two lines. In contrast, we have

Proposition 1.5. *If $d \neq 1, 0$, or -1, then $\mathbb{P}^2$ is not a biequivariant compactification of G_d.*

Proof. Assume otherwise. Let D_1 and D_2 be the two irreducible boundary components. Since $\mathcal{O}(K_{\mathbb{P}^2}) \cong \mathcal{O}(-3)$, either both components D_1 and D_2 are lines or one of them is a line and the other a conic. Let ω be a right invariant top degree differential form. Then $\omega/\varphi_d(a)$ is a left invariant differential form. If one of D_1 and D_2 is a conic, then the divisor of ω takes the form

$$-\mathrm{div}(\omega) = -\mathrm{div}(\omega/\varphi_d(a)) = D_1 + D_2,$$

but this is a contradiction. If D_1 and D_2 are lines, then without loss of generality, we can assume that

$$-\mathrm{div}(\omega) = 2D_1 + D_2 \quad \text{and} \quad -\mathrm{div}(\omega/\varphi_d(a)) = D_1 + 2D_2.$$

However, $\mathrm{div}(a)$ is a multiple of $D_1 - D_2$, which is also a contradiction. $\square$

Combining this result with Proposition 1.3, we conclude that a del Pezzo surface is not a biequivariant compactification of G_d, for $d \neq 1, 0, \mathrm{or}, -1$. Another sample result in this direction is:

Proposition 1.6. *Let S be the singular quartic del Pezzo surface of type $A_3 + A_1$ defined by*

$$x_0^2 + x_0x_3 + x_2x_4 = x_1x_3 - x_2^2 = 0$$

Then S is a one-sided equivariant compactification of G_1, but not a biequivariant compactification of G_d if $d \neq 0$.

Proof. For the first assertion, see [DL10, Section 5]. Assume that S is a biequivariant compactification of G_d. Let $\pi : \widetilde{S} \to S$ be its minimal desingularization. Then $\widetilde{S}$ is also a biequivariant compactification of G_d. It has three (-1)-curves L_1, L_2, and L_3, which are the strict transforms of

$$\{x_0 = x_1 = x_2 = 0\}, \quad \{x_0 + x_3 = x_1 = x_2 = 0\}, \quad \text{and} \quad \{x_0 = x_2 = x_3 = 0\},$$

respectively, and has four (-2)-curves R_1, R_2, R_3, and R_4. The nonzero intersection numbers are given by:

$$L_1.R_1 = L_2.R_1 = R_1.R_2 = R_2.R_3 = R_3.L_3 = L_3.R_4 = 1.$$

Since the cone of curves is generated by the components of the boundary, these negative curves must be in the boundary because each generates an extremal ray. Since the Picard group of $\widetilde{S}$ has rank six, the number of boundary components is seven. Thus, the boundary is equal to the union of these negative curves.

Let $f : \widetilde{S} \to \mathbb{P}^2$ be the birational morphism which contracts L_1, L_2, L_3, R_2, and R_3. This induces a biequivariant compactification on $\mathbb{P}^2$. The birational map $f \circ \pi^{-1} : S \dashrightarrow \mathbb{P}^2$ is given by $S \ni (x_0 : x_1 : x_2 : x_3 : x_4) \mapsto (x_2 : x_0 : x_3) \in \mathbb{P}^2$. The images of R_1 and R_4 are $\{y_0 = 0\}$ and $\{y_2 = 0\}$ and we denote them by D_0 and D_2, respectively. The images of L_1 and L_2 are $(0 : 0 : 1)$ and $(0 : 1 : -1)$, respectively; so that the induced group action on $\mathbb{P}^2$ must fix $(0 : 0 : 1)$, $(0 : 1 : -1)$, and $D_0 \cap D_2 = (0 : 1 : 0)$. Thus, the group action must fix the line D_0, and this fact implies that all left and right invariant vector fields vanish along D_0. It follows that

$$-\mathrm{div}(\omega) = -\mathrm{div}(\omega/\varphi_d(a)) = 2D_0 + D_2,$$

which contradicts $d \neq 0$. □

Example 1.7. Let $l \geq d \geq 0$. The Hirzebruch surface $\mathbb{F}_l = \mathbb{P}_{\mathbb{P}^1}((\mathcal{O} \oplus \mathcal{O}(l))^*)$ is a biequivariant compactification of G_d. Indeed, we may take the embedding

$$\begin{array}{ccc} G_d & \hookrightarrow & \mathbb{F}_l \\ (x, a) & \mapsto & ((a : 1), [1 \oplus x\sigma_1^l]), \end{array}$$

where σ_1 is a section of the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^1$ such that

$$\mathrm{div}(\sigma_1) = (1 : 0).$$

Let $\pi : \mathbb{F}_l \to \mathbb{P}^1$ be the $\mathbb{P}^1$-fibration. The right action is given by

$$((x_0 : x_1), [y_0 \oplus y_1\sigma_1^l]) \mapsto ((ax_0 : x_1), [y_0 \oplus (y_1 + (x_0/x_1)^d xy_0)\sigma_1^l]),$$

on $\pi^{-1}(U_0 = \mathbb{P}^1 \setminus \{(1 : 0)\})$ and

$$((x_0 : x_1), [y_0 \oplus y_1\sigma_0^l]) \mapsto ((ax_0 : x_1), [a^l y_0 \oplus (y_1 + (x_1/x_0)^{l-d} xy_0)\sigma_0^l]),$$

on $U_1 = \pi^{-1}(\mathbb{P}^1 \setminus \{(0 : 1)\})$. Similarly, one defines the left action. The boundary consists of three components: two fibers $f_0 = \pi^{-1}((0 : 1))$, $f_1 = \pi^{-1}((0 : 1))$ and the special section D characterized by $D^2 = -l$.

Example 1.8. Consider the right actions in Examples 1.7. When $l > d > 0$, these actions fix the fiber f_0 and act multiplicatively, i.e., with two fixed points, on the fiber f_1. Let X be the blow up of two points (or more) on f_0 and of one fixed point P on $f_1 \setminus D$. Then X is an equivariant compactification of G_d which is neither a toric variety nor a $\mathbb{G}_a^2$-variety. Indeed, there are no equivariant compactifications of $\mathbb{G}_m^2$ on $\mathbb{F}_l$ fixing f_0, so X cannot be toric. Also, if X were a $\mathbb{G}_a^2$-variety, we would obtain an induced $\mathbb{G}_a^2$-action on $\mathbb{F}_l$ fixing f_0 and P. However, the boundary consists of two irreducible components and must contain f_0, D, and P because D is a negative curve. This is a contradiction.

For $l = 2$ and $d = 1$, blowing up two points on f_0 we obtain a quintic del Pezzo surface with an A_2 singularity. Manin's conjecture for this surface is proved in [Der07].

In Section 5, we prove Manin's conjecture for X with $l \geq 3$.

2. Height zeta functions

Let F be a number field, $\mathfrak{o}_F$ its ring of integers, and Val_F the set of equivalence classes of valuations of F. For $v \in \mathrm{Val}_F$ let F_v be the completions of F with respect to v, for nonarchimedean v, let $\mathfrak{o}_v$ be the corresponding ring of integers and $\mathfrak{m}_v$ the maximal ideal. Let $\mathbb{A} = \mathbb{A}_F$ be the adele ring of F.

Let X be a smooth and projective right-sided equivariant compactification of a split connected solvable linear algebraic group G over F, i.e., the toric part T of G is isomorphic to $\mathbb{G}_m^n$. Moreover, we assume that the boundary $D = \cup_{\iota \in \mathcal{I}} D_\iota$ consists of geometrically irreducible components meeting transversely. We are interested in the asymptotic distribution of rational points of bounded height on $X^\circ = G \subset X$, with respect to adelically metrized ample line bundles $\mathcal{L} = (L, (\|\cdot\|_{\mathbb{A}}))$ on X. We now recall the method of height zeta functions; see [Tsc09, Section 6] for more details and examples.

Step 1. Define an adelic height pairing

$$\mathsf{H} \colon \mathrm{Pic}^G(X)_{\mathbb{C}} \times G(\mathbb{A}_F) \to \mathbb{C},$$

whose restriction to

$$\mathsf{H} \colon \mathrm{Pic}^G(X) \times G(F) \to \mathbb{R}_{\geq 0},$$

descends to a *height system* on $\mathrm{Pic}(X)$ (see [Pey98, Definition 2.5.2]). This means that the restriction of H to an $L \in \mathrm{Pic}^G(X)$ defines a Weil height corresponding to some adelic metrization of $L \in \mathrm{Pic}^G(X)$, and that it does not depend on the choice of a G-linearization on L. Such a pairing appeared in [BT95] in the context of toric varieties, the extension to general solvable groups is straightforward.

Concretely, by Proposition 1.1, we know that $\mathrm{Pic}^G(X)$ is generated by boundary components D_ι, for $\iota \in \mathcal{I}$. The v-adic analytic manifold $X(F_v)$ admits a "partition of unity", i.e., a decomposition into charts $X_{I,v}$, labeled by $I \subseteq \mathcal{I}$, such that in each chart the local height function takes the form

$$\mathsf{H}_v(\mathbf{s}, x_v) = \phi(x_v) \cdot \prod_{\iota \in I} |x_{\iota,v}|_v^{s_\iota},$$

where for each $\iota \in I$, x_ι is the local coordinate of D_ι in this chart,

$$\mathbf{s} = \sum_{\iota \in \mathcal{I}} s_\iota D_\iota,$$

and ϕ is a bounded function, equal to 1 for almost all v. Note that, locally, the height function

$$\mathsf{H}_{\iota,v}(x_v) := |x_{\iota,v}|_v$$

is simply the v-adic distance to the boundary component D_ι. To visualize $X_{I,v}$ (for almost all v) consider the partition induced by

$$X(F_v) = X(\mathfrak{o}_v) \xrightarrow{\rho} \sqcup_{I \subset \mathcal{I}} X_I^\circ(\mathbb{F}_q),$$

where

$$X_I := \cup_{\iota \in I} D_I, \quad X_I^\circ := X_I \setminus \cup_{I' \supsetneq I} X_{I'},$$

is the stratification of the boundary and ρ is the reduction map; by convention $X_\emptyset = G$. Then $X_{I,v}$ is the preimage of $X_I^\circ(\mathbb{F}_q)$ in $X(F_v)$, and in particular, $X_{\emptyset,v} = G(\mathfrak{o}_v)$, for almost all v.

Since the action of G lifts to integral models of G, X, and L, the nonarchimedean local height pairings are invariant with respect to a compact subgroup $\mathbf{K}_v \subset G(F_v)$, which is $G(\mathfrak{o}_v)$, for almost all v.

Step 2. The height zeta function

$$\mathsf{Z}(\mathbf{s}, g) := \sum_{\gamma \in G(F)} \mathsf{H}(\mathbf{s}, \gamma g)^{-1},$$

converges absolutely to a holomorphic function, for $\Re(\mathbf{s})$ sufficiently large, and defines a continuous function in $\mathsf{L}^1(G(F)\backslash G(\mathbb{A}_F)) \cap \mathsf{L}^2(G(F)\backslash G(\mathbb{A}_F))$. Formally, we have the spectral expansion

$$\mathsf{Z}(\mathbf{s}, g) = \sum_\pi \mathsf{Z}_\pi(\mathbf{s}, g), \tag{2.1}$$

where the "sum" is over irreducible unitary representations occurring in the right regular representation of $G(\mathbb{A}_F)$ in $\mathsf{L}^2(G(F)\backslash G(\mathbb{A}_F))$. The invariance of the global height pairing under the action of a compact subgroup $\mathbf{K} \subset G(\mathbb{A}_F)$, on the side of the action, insures that Z_π are in $\mathsf{L}^2(G(F)\backslash G(\mathbb{A}_F))^{\mathbf{K}}$.

Step 3. Ideally, we would like to obtain a meromorphic continuation of Z to a tube domain

$$\mathsf{T}_\Omega = \Omega + i \operatorname{Pic}(X)_\mathbb{R} \subset \operatorname{Pic}(X)_\mathbb{C},$$

where $\Omega \subset \operatorname{Pic}(X)_\mathbb{R}$ is an open neighborhood of the anticanonical class $-K_X$. It is expected that Z is holomorphic for

$$\Re(\mathbf{s}) \in -K_X + \Lambda_{\mathrm{eff}}^\circ(X)$$

and that the polar set of the shifted height zeta function $\mathsf{Z}(\mathbf{s} - K_X, g)$ is the same as that of

$$\mathcal{X}_{\Lambda_{\mathrm{eff}}(X)}(\mathbf{s}) := \int_{\Lambda_{\mathrm{eff}}^*(X)} e^{-\langle \mathbf{s}, \mathbf{y} \rangle} \mathrm{d}y, \tag{2.2}$$

the Laplace transform of the set-theoretic characteristic function of the dual cone $\Lambda_{\mathrm{eff}}(X)^* \subset \operatorname{Pic}(X)_\mathbb{R}^*$. Here the Lebesgue measure $\mathrm{d}y$ is normalized by the dual lattice $\operatorname{Pic}(X)^* \subset \operatorname{Pic}(X)_\mathbb{R}^*$. In particular, for

$$\kappa = -K_X = \sum_\iota \kappa_\iota D_\iota,$$

the restriction of the height zeta function $\mathsf{Z}(\mathbf{s}, \mathrm{id})$ to the one-parameter zeta function $\mathsf{Z}(s\kappa, \mathrm{id})$ should be holomorphic for $\Re(s) > 1$, admit a meromorphic continuation to $\Re(s) > 1-\epsilon$, for some $\epsilon > 0$, with a unique pole at $s = 1$, of order $r = \mathrm{rk}\,\mathrm{Pic}(X)$. Furthermore, is is desirable to have some growth estimates in vertical strips. In this case, a Tauberian theorem implies Manin's conjecture (0.1) for the counting function; the quality of the error term depends on the growth rate in vertical strips. Finally, the leading constant at the pole of $\mathsf{Z}(s\kappa, \mathrm{id})$ is essentially the Tamagawa-type number defined by Peyre. We will refer to this by saying that the height zeta function Z satisfies Manin's conjecture; a precise definition of this class of functions can be found in [CLT01, Section 3.1].

This strategy has worked well and lead to a proof of Manin's conjecture for the following varieties:

- toric varieties [BT95], [BT98], [BT96a];
- equivariant compactifications of additive groups $\mathbb{G}_a^n$ [CLT02];
- equivariant compactifications of unipotent groups [ST04], [ST];
- wonderful compactifications of semi-simple groups of adjoint type [STBT07].

Moreover, applications of Langlands' theory of Eisenstein series allowed to prove Manin's conjecture for flag varieties [FMT89], their twisted products [Str01], and horospherical varieties [ST99], [CLT01].

The analysis of the spectral expansion (2.1) is easier when every automorphic representation π is 1-dimensional, i.e., when G is abelian: $G = \mathbb{G}_a^n$ or $G = T$, an algebraic torus. In these cases, (2.1) is simply the Fourier expansion of the height zeta function and we have, at least formally,

$$\mathsf{Z}(\mathbf{s}, \mathrm{id}) = \int \widehat{\mathsf{H}}(\mathbf{s}, \chi)\mathrm{d}\chi, \tag{2.3}$$

where

$$\hat{\mathsf{H}}(\mathbf{s}, \chi) = \int_{G(\mathbb{A}_F)} \mathsf{H}(\mathbf{s}, g)^{-1}\bar{\chi}(g)\mathrm{d}g, \tag{2.4}$$

is the Fourier transform of the height function, χ is a character of $G(F)\backslash G(\mathbb{A}_F)$, and $\mathrm{d}\chi$ an appropriate measure on the space of automorphic characters. For $G = \mathbb{G}_a^n$, the space of automorphic characters is $G(F)$ itself, for G an algebraic torus it is (noncanonically) $\mathfrak{X}(G)_{\mathbb{R}}^{\times} \times \mathcal{U}_G$, where $\mathcal{U}_G$ is a discrete group.

The v-adic integration technique developed by Denef and Loeser (see [DL98], [DL99], and [DL01]) allows to compute local Fourier transforms of height functions, in particular, for the trivial character $\chi = 1$ and almost all v we obtain

$$\widehat{\mathsf{H}}_v(\mathbf{s}, 1) = \int_{G(F_v)} \mathsf{H}(\mathbf{s}, g)^{-1}\mathrm{d}g = \tau_v(G)^{-1}\left(\sum_{I\subset\mathcal{I}} \frac{\#X_I^{\circ}(\mathbb{F}_q)}{q^{\dim(X)}} \prod_{\iota\in I} \frac{q-1}{q^{s_\iota-\kappa_\iota+1}-1}\right),$$

where X_I are strata of the stratification described in Step 1 and $\tau_v(G)$ is the local Tamagawa number of G,

$$\tau_v(G) = \frac{\#G(\mathbb{F}_q)}{q^{\dim(G)}}.$$

Such height integrals are geometric versions of Igusa's integrals; a comprehensive theory in the analytic and adelic setting can be found in [CLT10].

The computation of Fourier transforms at nontrivial characters requires a finer partition of $X(F_v)$ which takes into account possible zeroes of the *phase* of the

character in $G(F_v)$; see [CLT02, Section 10] for the the additive case and [BT95, Section 2] for the toric case. The result is that in the neighborhood of

$$\kappa = \sum_{\iota} \kappa_\iota D_\iota \in \mathrm{Pic}^G(X),$$

the Fourier transform is regularized as follows

$$\widehat{\mathsf{H}}(\mathbf{s},\chi) = \begin{cases} \prod_{v\notin S(\chi)} \prod_{\iota\in\mathcal{I}(\chi)} \zeta_{F,v}(s_\iota - \kappa_\iota + 1) \prod_{v\in S(\chi)} \phi_v(\mathbf{s},\chi) & G = \mathbb{G}_a^n, \\ \prod_{v\notin S(\chi)} \prod_{\iota\in\mathcal{I}} \mathsf{L}_{F,v}(s_\iota - \kappa_\iota + 1 + im(\chi), \chi_u) \prod_{v\in S(\chi)} \phi_v(\mathbf{s},\chi) & G = T, \end{cases}$$

where

- $\mathcal{I}(\chi) \subsetneq \mathcal{I}$;
- $S(\chi)$ is a finite set of places, which, in general, depends on χ;
- $\zeta_{F,v}$ is a local factor of the Dedekind zeta function of F and $\mathsf{L}_{F,v}$ a local factor of a Hecke L-function;
- $m(\chi)$ is the "coordinate" of the automorphic character χ of $G = T$ under the embedding $\mathfrak{X}(G)^\times_{\mathbb{R}} \hookrightarrow \mathrm{Pic}^G(X)_{\mathbb{R}}$ in the exact sequence (1) in Proposition 1.1 and χ_u is the "discrete" component of χ;
- and $\phi_v(\mathbf{s},\chi)$ is a function which is holomorphic and bounded.

In particular, each $\widehat{\mathsf{H}}(\mathbf{s},\chi)$ admits a meromorphic continuation as desired and we can control the poles of each term. Moreover, at archimedean places we may use integration by parts with respect to vector fields in the universal enveloping algebra of the corresponding real of complex group to derive bounds in terms of the "phase" of the occurring oscillatory integrals, i.e., in terms of "coordinates" of χ.

So far, we have not used the fact that X is an equivariant compactification of G. Only at this stage do we see that the $\mathbf{K}$-invariance of the height is an important, in fact, crucial, property that allows to establish uniform convergence of the right side of the expansion (2.1); it insures that

$$\widehat{\mathsf{H}}(\mathbf{s},\chi) = 0,$$

for all χ which are *nontrivial* on $\mathbf{K}$. For $G = \mathbb{G}_a^n$ this means that the trivial representation is *isolated* and that the integral on the right side of Equation (2.3) is in fact a *sum* over a *lattice* of integral points in $G(F)$. Note that Manin's conjecture *fails* for nonequivariant compactifications of the affine space, there are counterexamples already in dimension three [BT96b]. The analytic method described above fails precisely because we cannot insure the convergence on the Fourier expansion.

A similar effect occurs in the noncommutative setting; one-sided actions *do not* guarantee bi-$\mathbf{K}$-invariance of the height, in contrast with the abelian case. Analytically, this translates into subtle convergence issues of the spectral expansion, in particular, for infinite-dimensional representation.

Theorem 2.1. *Let G be an extension of an algebraic torus T by a unipotent group N such that $[G,G] = N$ over a number field F. Let X be an equivariant compactification of G over F and*

$$\mathsf{Z}(\mathbf{s},g) = \sum_{\gamma\in G(F)} \mathsf{H}(\mathbf{s},\gamma g)^{-1},$$

the height zeta function with respect to an adelic height pairing as in Step 1. Let

$$\mathsf{Z}_0(\mathbf{s},g) = \int \mathsf{Z}_\chi(\mathbf{s},g)\,\mathrm{d}\chi,$$

be the integral over all 1-dimensional automorphic representations of $G(\mathbb{A}_F)$ *occurring in the spectral expansion* (2.1). *Then* Z_0 *satisfies Manin's conjecture.*

Proof. Let

$$1 \to N \to G \to T \to 1,$$

the defining extension. One-dimensional automorphic representations of $G(\mathbb{A}_F)$ are precisely those which are trivial on $N(\mathbb{A}_F)$, i.e., these are automorphic characters of T. The $\mathbf{K}$-invariance of the height (on one side) insures that only unramified characters, i.e., $\mathbf{K}_T$-invariant characters contribute to the spectral expansion of Z_0.

Let $M = \mathfrak{X}(G)^\times$ be the group of algebraic characters. We have

$$\begin{aligned}
\mathsf{Z}_0(\mathbf{s}, \mathrm{id}) &= \int_{M_\mathbb{R} \times \mathcal{U}_T} \int_{G(F)\backslash G(\mathbb{A}_F)} \mathsf{Z}(\mathbf{s}, g) \bar{\chi}(g) \, \mathrm{d}g \mathrm{d}\chi \\
&= \int_{M_\mathbb{R} \times \mathcal{U}_T} \int_{G(\mathbb{A}_F)} \mathsf{H}(\mathbf{s}, g)^{-1} \bar{\chi}(g) \, \mathrm{d}g \mathrm{d}\chi \\
&= \int_{M_\mathbb{R}} \mathsf{F}(\mathbf{s} + im(\chi)) \, \mathrm{d}m,
\end{aligned}$$

where

$$\mathsf{F}(\mathbf{s}) := \sum_{\chi \in \mathcal{U}_T} \widehat{\mathsf{H}}(\mathbf{s}, \chi_u).$$

Computations of local Fourier transforms explained above show that F can be regularized as follows:

$$\mathsf{F}(\mathbf{s}) = \prod_{\iota \in \mathcal{I}} \zeta_F(s_\iota - \kappa_\iota + 1) \cdot \mathsf{F}_\infty(\mathbf{s}),$$

where F_∞ is holomorphic for $\Re(s_\iota) - \kappa_\iota > -\epsilon$, for some $\epsilon > 0$, with growth control in vertical strips. Now we have placed ourselves into the situation considered in [CLT01, Section 3]: Theorem 3.1.14 establishes analytic properties of integrals

$$\int_{M_\mathbb{R}} \frac{1}{\prod_{\iota \in \mathcal{I}} (s_\iota - \kappa_\iota + im_\iota)} \cdot \mathsf{F}_\infty(\mathbf{s} + im) \, \mathrm{d}m,$$

where the image of $\iota\colon M_\mathbb{R} \hookrightarrow \mathbb{R}^{\#\mathcal{I}}$ intersects the simplicial cone $\mathbb{R}^{\#\mathcal{I}}_{\geq 0}$ only in the origin. The main result is that the analytic properties of such integrals match those of the $\mathcal{X}$-function (2.2) of the *image* cone under the projection

$$\begin{tikzcd}
0 \arrow[r] & M_\mathbb{R} \arrow[r, "\iota"] \arrow[d, equal] & \mathbb{R}^{\#\mathcal{I}} \arrow[r, "\pi"] \arrow[d, equal] & \mathbb{R}^{\#\mathcal{I} - \dim(M)} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{X}(G)^\times_\mathbb{R} \arrow[r] & \mathrm{Pic}^G(X)_\mathbb{R} \arrow[r] & \mathrm{Pic}(X)_\mathbb{R} \arrow[r] & 0;
\end{tikzcd}$$

according to Proposition 1.1, the image of the simplicial cone $\mathbb{R}^{\#\mathcal{I}}_{\geq 0}$ under π is precisely $\Lambda_{\mathrm{eff}}(X) \subset \mathrm{Pic}(X)_\mathbb{R}$. ∎

3. Harmonic analysis

In this section we study the local and adelic representation theory of

$$G := \mathbb{G}_a \rtimes_\varphi \mathbb{G}_m,$$

an extension of $T := \mathbb{G}_m$ by $N := \mathbb{G}_a$ via a homomorphism $\varphi : \mathbb{G}_m \to \mathrm{GL}_1$. The group law given by

$$(x, a) \cdot (y, b) = (x + \varphi(a)y, ab).$$

We fix the standard Haar measures

$$\mathrm{d}x = \prod_v \mathrm{d}x_v \quad \text{and} \quad \mathrm{d}a^\times = \prod_v \mathrm{d}a_v^\times,$$

on $N(\mathbb{A}_F)$ and $T(\mathbb{A}_F)$. Note that $G(\mathbb{A}_F)$ is not unimodular; $\mathrm{d}g := \mathrm{d}x\mathrm{d}a^\times$ is a right invariant measure on $G(\mathbb{A}_F)$ and $\mathrm{d}g/\varphi(a)$ is a left invariant measure.

Let ϱ be the right regular unitary representation of $G(\mathbb{A}_F)$ on the Hilbert space:

$$\mathcal{H} := \mathsf{L}^2(G(F)\backslash G(\mathbb{A}_F), \mathrm{d}g).$$

We now discuss the decomposition of $\mathcal{H}$ into irreducible representations. Let

$$\psi = \prod_v \psi_v : \mathbb{A}_F \to \mathbb{S}^1,$$

be the standard automorphic character and ψ_n the character defined by

$$x \to \psi(nx),$$

for $n \in F^\times$. Let

$$W := \ker(\varphi : F^\times \to F^\times),$$

and

$$\pi_n := \mathrm{Ind}_{N(\mathbb{A}_F)\times W}^{G(\mathbb{A}_F)}(\psi_n),$$

for $n \in F^\times$. The following proposition we learned from J. Shalika [Sha].

Proposition 3.1. *Irreducible automorphic representations, i.e., irreducible unitary representations occurring in $\mathcal{H} = \mathsf{L}^2(G(F)\backslash G(\mathbb{A}_F))$, are parametrized as follows:*

$$\mathcal{H} = \mathsf{L}^2(T(F)\backslash T(\mathbb{A}_F)) \oplus \widehat{\bigoplus}_{n\in(F^\times/\varphi(F^\times))} \pi_n,$$

Remark 3.2. Up to unitary equivalence, the representation π_n does not depend on the choice of a representative $n \in F^\times/\varphi(F^\times)$.

Proof. Define

$$\mathcal{H}_0 := \{\phi \in \mathcal{H} \,|\, \phi((x, 1)g) = \phi(g)\},$$

and let $\mathcal{H}_1$ be the orthogonal complement of $\mathcal{H}_0$. It is straightforward to prove that

$$\mathcal{H}_0 \cong \mathsf{L}^2(T(F)\backslash T(\mathbb{A}_F)).$$

The following two lemmas prove that

$$\mathcal{H}_1 \cong \widehat{\bigoplus}_{n\in F^\times/\varphi(F^\times)} \pi_n.$$

□

Lemma 3.3. *For any $\phi \in \mathsf{L}^1(G(F)\backslash G(\mathbb{A}_F)) \cap \mathcal{H}$, the projection of ϕ onto $\mathcal{H}_0$ is given by*

$$\phi_0(g) := \int_{N(F)\backslash N(\mathbb{A}_F)} \phi((x, 1)g)dx.$$

Proof. It is easy to check that $\phi_0 \in \mathcal{H}_0$. Also, for any $\phi' \in \mathcal{H}_0$, we have

$$\int_{G(F)\backslash G(\mathbb{A}_F)} (\phi - \phi_0)\phi' \mathrm{d}g = 0.$$

□

Lemma 3.4. *We have*

$$\mathcal{H}_1 \cong \widehat{\bigoplus}_{n \in F^\times/\varphi(F^\times)} \pi_n.$$

Proof. First we note that the underlying Hilbert space of π_n is $\mathsf{L}^2(W\backslash T(\mathbb{A}_F))$, and that the group action is given by

$$(x, a) \cdot f(b) = \psi_n(\varphi(b)x) f(ab),$$

where f is a square-integrable function on $T(\mathbb{A}_F)$. For $\phi \in C_c^\infty(G(F)\backslash G(\mathbb{A}_F)) \cap \mathcal{H}_1$, define

$$f_{n,\phi}(a) := \int_{N(F)\backslash N(\mathbb{A}_F)} \phi(x, a)\overline{\psi}_n(x)\, \mathrm{d}x.$$

Then,

$$\begin{aligned}
\| \phi \|_{\mathsf{L}^2}^2 &= \int_{T(F)\backslash T(\mathbb{A}_F)} \int_{N(F)\backslash N(\mathbb{A}_F)} |\phi(x, a))|^2 \,\mathrm{d}x \mathrm{d}a^\times \\
&= \int_{T(F)\backslash T(\mathbb{A}_F)} \sum_{\alpha \in F} \left| \int_{N(F)\backslash N(\mathbb{A}_F)} \phi(x, a)\overline{\psi}(\alpha x)\mathrm{d}x \right|^2 \mathrm{d}a^\times \\
&= \int_{T(F)\backslash T(\mathbb{A}_F)} \sum_{\alpha \in F^\times} \left| \int_{N(F)\backslash N(\mathbb{A}_F)} \phi(x, a)\overline{\psi}(\alpha x)\mathrm{d}x \right|^2 \mathrm{d}a^\times \\
&= \int_{T(F)\backslash T(\mathbb{A}_F)} \sum_{\alpha \in F^\times} \sum_{n \in F^\times/\varphi(F^\times)} \frac{1}{\#W} \left| \int_{N(F)\backslash N(\mathbb{A}_F)} \phi(x, a)\overline{\psi}(n\varphi(\alpha)x)\mathrm{d}x \right|^2 \mathrm{d}a^\times \\
&= \int_{T(F)\backslash T(\mathbb{A}_F)} \sum_{\alpha \in F^\times} \sum_{n \in F^\times/\varphi(F^\times)} \frac{1}{\#W} \left| \int_{N(F)\backslash N(\mathbb{A}_F)} \phi(x, \alpha a)\overline{\psi}(nx)\mathrm{d}x \right|^2 \mathrm{d}a^\times \\
&= \sum_{n \in F^\times/\varphi(F^\times)} \frac{1}{\#W} \int_{T(\mathbb{A}_F)} \left| \int_{N(F)\backslash N(\mathbb{A}_F)} \phi(x, a)\overline{\psi}_n(x)dx \right|^2 \mathrm{d}a^\times \\
&= \sum_{n \in F^\times/\varphi(F^\times)} \| f_{n,\phi} \|_{\mathsf{L}^2}^2 .
\end{aligned}$$

The second equality is the Plancherel theorem for $N(F)\backslash N(\mathbb{A}_F)$. Third equality follows from the previous lemma. The fourth equality follows from the left $G(F)$-invariance of ϕ. Thus, we obtain an unitary operator:

$$I : \mathcal{H}_1 \to \widehat{\bigoplus}_{n \in F^\times/\varphi(F^\times)} \pi_n.$$

Compatibility with the group action is straightforward, so I is actually a morphism of unitary representations. We construct the inverse map of I explicitly. For $f \in \mathsf{C}_c^\infty(W\backslash T(\mathbb{A}_F))$, define

$$\phi_{n,f}(x, a) := \frac{1}{\#W} \sum_{\alpha \in F^\times} \psi_n(\varphi(\alpha)x) f(\alpha a).$$

The orthogonality of characters implies that

$$\int_{N(F)\backslash N(\mathbb{A}_F)} \phi_{n,f}(x,a)\cdot\overline{\phi_{n,f}(x,a)}\,\mathrm{d}x$$

$$= \int_{N(F)\backslash N(\mathbb{A}_F)} (\textstyle\sum_{\alpha\in F^\times}\psi_n(\varphi(\alpha)x)f(\alpha a))\cdot(\sum_{\alpha\in F^\times}\overline{\psi}_n(\varphi(\alpha)x)\overline{f}(\alpha a))\,\mathrm{d}x$$

$$= \textstyle\sum_{\alpha\in F^\times}|f(\alpha a)|^2.$$

Substituting, we obtain

$$\begin{aligned}\|\ \phi_{n,f}\ \|^2 &= \int_{T(F)\backslash T(\mathbb{A}_F)}\int_{N(F)\backslash N(\mathbb{A}_F)} |\phi_{n,f}(x,a)|^2\,\mathrm{d}x\mathrm{d}a^\times \\ &= \frac{1}{\#W}\int_{T(F)\backslash T(\mathbb{A}_F)}\sum_{\alpha\in F^\times}|f(\alpha a)|^2\mathrm{d}a^\times = \|\ f\ \|_n^2\ .\end{aligned}$$

Lemma 3.3 implies that that $\phi_f\in\mathcal{H}_1$ and we obtain a morphism

$$\Theta : \widehat{\bigoplus}_{n\in F^\times/\varphi(F^\times)}\pi_n \to \mathcal{H}_1.$$

Now we only need to check that $\Theta I = id$ and $I\Theta = id$. The first follows from the Poisson formula: For any $\phi\in\mathsf{C}_c^\infty(G(F)\backslash G(\mathbb{A}_F))\cap\mathcal{H}_1$,

$$\begin{aligned}\Theta I\phi &= \sum_{n\in F^\times/\varphi(F^\times)}\frac{1}{\#W}\sum_{\alpha\in F^\times}\psi_n(\varphi(\alpha)x)\int_{N(F)\backslash N(\mathbb{A}_F)}\phi(y,\alpha a)\overline{\psi}_n(y)\mathrm{d}y \\ &= \sum_{n\in F^\times/\varphi(F^\times)}\frac{1}{\#W}\sum_{\alpha\in F^\times}\int_{N(F)\backslash N(\mathbb{A}_F)}\phi(\varphi(\alpha)y,\alpha a)\overline{\psi}_n(\varphi(\alpha)(y-x))\mathrm{d}y \\ &= \sum_{n\in F^\times/\varphi(F^\times)}\frac{1}{\#W}\sum_{\alpha\in F^\times}\int_{N(F)\backslash N(\mathbb{A}_F)}\phi((y+x,a))\overline{\psi}_n(\varphi(\alpha)y)\mathrm{d}y \\ &= \sum_{\alpha\in F}\int_{N(F)\backslash N(\mathbb{A}_F)}\phi((y,1)(x,a))\overline{\psi}(\alpha y)\mathrm{d}y = \phi(x,a).\end{aligned}$$

The other identity, $I\Theta = id$ is checked by a similar computation. □

To simplify notation, we now restrict to $F=\mathbb{Q}$. For our applications in Sections 4 and 5, we need to know an explicit orthonormal basis for the unique infinite-dimensional representation $\pi = \mathsf{L}^2(\mathbb{A}_\mathbb{Q}^\times)$ of $G = G_1$. For any $n\geq 1$, define compact subgroups of $G(\mathbb{Z}_p)$

$$G(p^n\mathbb{Z}_p) := \{(x,a)\,|\,x\in p^n\mathbb{Z}_p,\ a\in 1+p^n\mathbb{Z}_p\}.$$

Let $v_p:\mathbb{Q}_p\to\mathbb{Z}$ be the discrete valuation on $\mathbb{Q}_p$.

Lemma 3.5. *Let $\mathbf{K}_p = G(p^n\mathbb{Z}_p)$.*

- *When $n=0$, an orthonormal basis for $\mathsf{L}^2(\mathbb{Q}_p^\times)^{\mathbf{K}_p}$ is given by*

$$\{\mathbf{1}_{p^j\mathbb{Z}_p^\times}\,|\,j\geq 0\}.$$

- *When $n\geq 1$, an orthonormal basis for $\mathsf{L}^2(\mathbb{Q}_p^\times)^{\mathbf{K}_p}$ is given by*

$$\{\lambda_p(\cdot/p^j)\mathbf{1}_{p^j\mathbb{Z}_p^\times}\,|\,j\geq -n,\ \lambda_p\in\mathsf{M}_p\},$$

where M_p is the set of multiplicative characters on $\mathbb{Z}_p^\times/(1+p^n\mathbb{Z}_p)$.

Moreover, let $\mathbf{K}_{\mathrm{fin}} = \prod_p \mathbf{K}_p$ *where* $\mathbf{K}_p = G(p^{n_p}\mathbb{Z}_p)$ *and* $n_p = 0$ *for almost all* p. *Let* S *be the set of primes with* $n_p \neq 0$ *and* $N = \prod_p p^{n_p}$. *Then an orthonormal basis for* $\mathsf{L}^2(\mathbb{A}^\times_{\mathbb{Q},\mathrm{fin}})^{\mathbf{K}_{\mathrm{fin}}}$ *is given by*

$$\{\otimes_{p\in S}\lambda_p(a_p \cdot p^{-v_p(a_p)})\mathbf{1}_{\frac{m}{N}\mathbb{Z}_p^\times}(a_p) \otimes'_{p\notin S} \mathbf{1}_{m\mathbb{Z}_p^\times}(a_p) \,|\, m \in \mathbb{N},\, \lambda_p \in \mathsf{M}_p\}.$$

Proof. For the first assertion, let $f \in \mathsf{L}^2(\mathbb{Q}_p^\times)^{\mathbf{K}_p}$ where $\mathbf{K}_p = G(\mathbb{Z}_p)$. Since it is $\mathbf{K}_p$-invariant, we have

$$f(b_p \cdot a_p) = f(a_p),$$

for any $b \in \mathbb{Z}_p^\times$. Hence f takes the form of

$$f = \sum_{j=-\infty}^{\infty} c_j \mathbf{1}_{p^j\mathbb{Z}_p^\times}$$

where $c_j = f(p^j)$ and $\sum_{j=-\infty}^{\infty} |c_j|^2 < +\infty$. On the other hand we have

$$\psi_p(a_p \cdot x_p) f(a_p) = f(a_p)$$

for any $x_p \in \mathbb{Z}_p$. This implies that $f(p^j) = 0$ for any $j < 0$. Thus the first assertion follows. The second assertion is treated similarly. The last assertion follows from the first and the second assertions. □

We denote these vectors by $\mathbf{v}_{m,\lambda}$ where $m \in \mathbb{N}$ and $\lambda \in \mathsf{M} := \prod_{p\in S} \mathsf{M}_p$. Note that M is a finite set. Also we define

$$\begin{aligned}\theta_{m,\lambda,t}(g) &:= \Theta(\mathbf{v}_{m,\lambda} \otimes |\cdot|_\infty^{it})(g)\\ &= \sum_{\alpha\in\mathbb{Q}^\times} \psi(\alpha x)\mathbf{v}_{m,\lambda}(\alpha a_{\mathrm{fin}})\, |\alpha a_\infty|_\infty^{it}.\end{aligned}$$

The following proposition is a combination of Lemma 3.5 and the standard Fourier analysis on the real line:

Proposition 3.6. *Let* $f \in \mathcal{H}_1^{\mathbf{K}}$. *Suppose that*

(1) $I(f)$ *is integrable, i.e.,*

$$I(f) \in \mathsf{L}^2(\mathbb{A}^\times)^{\mathbf{K}} \cap \mathsf{L}^1(\mathbb{A}^\times),$$

(2) *the Fourier transform of* f *is also integrable i.e.*

$$\int_{-\infty}^{+\infty} |(f, \theta_{m,\lambda,t})|\,\mathrm{d}t < +\infty,$$

for any $m \in \mathbb{N}$ *and* $\lambda \in \mathbf{M}$.

Then we have

$$f(g) = \sum_{\lambda\in\mathbf{M}} \sum_{m=1}^{\infty} \frac{1}{4\pi} \int_{-\infty}^{+\infty} (f, \theta_{m,\lambda,t})\theta_{m,\lambda,t}(g)\,\mathrm{d}t \quad \text{a.e.},$$

where

$$(f, \theta_{m,\lambda,t}) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}})} f(g)\overline{\theta}_{m,\lambda,t}(g)\,\mathrm{d}g.$$

Proof. For simplicity, we assume that $n_p = 0$ for all primes p. Let $I(f) = h \in \mathsf{L}^2(\mathbb{A}^\times)^{\mathbf{K}} \cap \mathsf{L}^1(\mathbb{A}^\times)$. It follows from the proof of Lemma 3.4 that

$$f(g) = \Theta(h)(g) = \sum_{\alpha \in \mathbb{Q}^\times} \psi(\alpha x) h(\alpha a).$$

Note that this infinite sum exists in both L^1 and L^2 sense. It is easy to check that

$$\int_{\mathbb{A}^\times} h(a)\mathbf{v}_m(a_{\mathrm{fin}})|a_\infty|_\infty^{-it}\,\mathrm{d}a^\times = (f, \theta_{m,t}).$$

Write

$$h = \sum_m \mathbf{v}_m \otimes h_m,$$

where $h_m \in \mathsf{L}^2(\mathbb{R}_{>0}, \mathrm{d}a_\infty^\times)$. The first and the second assumptions imply that h_m and the Fourier transform of $\widehat{h}_m$ both are integrable. Hence the inverse formula of Fourier transformation on the real line implies that

$$h(a) = \sum_m \frac{1}{4\pi} \int_{-\infty}^{+\infty} (f, \theta_{m,t})\mathbf{v}_m(a_{\mathrm{fin}})|a_\infty|_\infty^{it}\,\mathrm{d}t \quad \text{a.e..}$$

Apply Θ to both sides, and our assertion follows. □

We recall some results regarding Igusa integrals with rapidly oscillating phase, studied in [CLT09]:

Proposition 3.7. *Let p be a finite place of $\mathbb{Q}$ and $d, e \in \mathbb{Z}$. Let*

$$\Phi : \mathbb{Q}_p^2 \times \mathbb{C}^2 \to \mathbb{C},$$

be a function such that for each $(x, y) \in \mathbb{Q}_p^2$, $\Phi((x, y), \mathbf{s})$ is holomorphic in $\mathbf{s} = (s_1, s_2) \in \mathbb{C}^2$. Assume that the function $(x, y) \mapsto \Phi(x, y, \mathbf{s})$ belongs to a bounded subset of the space of smooth compactly supported functions when $\Re(\mathbf{s})$ belongs to a fixed compact subset of $\mathbb{R}^2$. Let Λ be the interior of a closed convex cone generated by

$$(1, 0), (0, 1), (d, e).$$

Then, for any $\alpha \in \mathbb{Q}_p^\times$,

$$\eta_\alpha(\mathbf{s}) = \int_{\mathbb{Q}_p^2} |x|_p^{s_1}|y|_p^{s_2}\psi_p(\alpha x^d y^e)\Phi(x, y, \mathbf{s})\mathrm{d}x_p^\times \mathrm{d}y_p^\times,$$

is holomorphic on T_Λ. The same argument holds for the infinite place when Φ is a smooth function with compact supports.

Proof. For the infinite place, use integration by parts and apply the convexity principle. For finite places, assume that d, e are both negative. Let $\delta(x, y) = 1$ if $|x|_p = |y|_p = 1$ and 0 else. Then we have

$$\begin{aligned}\eta_\alpha(\mathbf{s}) &= \sum_{n,m \in \mathbb{Z}} \int_{\mathbb{Q}_p^2} |x|_p^{s_1}|y|_p^{s_2}\psi_p(\alpha x^d y^e)\Phi(x, y, \mathbf{s})\delta(p^{-n}x, p^{-m}y)\,\mathrm{d}x_p^\times \mathrm{d}y_p^\times \\ &= \sum_{n,m \in \mathbb{Z}} p^{-(ns_1 + ms_2)} \cdot \eta_{\alpha,n,m}(\mathbf{s}),\end{aligned}$$

where

$$\eta_{\alpha,n,m}(\mathbf{s}) = \int_{|x|_p = |y|_p = 1} \psi_p(\alpha p^{nd+me} x^d y^e)\Phi(p^n x, p^m y, \mathbf{s})\,\mathrm{d}x_p^\times \mathrm{d}y_p^\times.$$

Fix a compact subset of $\mathbb{C}^2$ and assume that $\Re(\mathbf{s})$ is in that compact set. The assumptions in our proposition mean that the support of $\Phi(\cdot, \mathbf{s})$ is contained in a fixed compact set in $\mathbb{Q}_p^2$, so there exists an integer N_0 such that $\eta_{\alpha,n,m}(\mathbf{s}) = 0$ if $n < N_0$ or $m < N_0$. Moreover our assumptions imply that there exists a positive real number δ such that $\Phi(\cdot, \mathbf{s})$ is constant on any ball of radius δ in $\mathbb{Q}_p^2$. This implies that if $1/p^n < \delta$, then for any $u \in \mathbb{Z}_p^\times$,

$$\begin{aligned}
\eta_{\alpha,n,m}(\mathbf{s}) &= \int \psi_p(\alpha p^{nd+me} x^d y^e u^d)\Phi(p^n xu, p^m y, \mathbf{s})\, \mathrm{d}x_p^\times \mathrm{d}y_p^\times \\
&= \int \psi_p(\alpha p^{nd+me} x^d y^e u^d)\Phi(p^n x, p^m y, \mathbf{s})\, \mathrm{d}x_p^\times \mathrm{d}y_p^\times \\
&= \int \int_{\mathbb{Z}_p^\times} \psi_p(\alpha p^{nd+me} x^d y^e u^d)\, \mathrm{d}u^\times\, \Phi(p^n x, p^m y, \mathbf{s})\, \mathrm{d}x_p^\times \mathrm{d}y_p^\times,
\end{aligned}$$

and the last integral is zero if n is sufficiently large because of [CLT09, Lemma 2.3.5]. Thus we conclude that there exists an integer N_1 such that $\eta_{\alpha,n,m}(\mathbf{s}) = 0$ if $n > N_1$ or $m > N_1$. Hence we obtained that

$$\eta_\alpha(\mathbf{s}) = \sum_{N_0 \leq n,m \leq N_1} p^{-(ns_1 + ms_2)} \cdot \eta_{\alpha,n,m}(\mathbf{s}),$$

and this is holomorphic everywhere.

The case of $d < 0$ and $e = 0$ is treated similarly.

Next assume that $d < 0$ and $e > 0$. Then again we have a constant c such that $\eta_{\alpha,n,m}(\mathbf{s}) = 0$ if $1/p^n < \delta$ and $n|d| - me > c$. We may assume that c is sufficiently large so that the first condition is unnecessary. Then we have

$$\begin{aligned}
|\eta_\alpha(\mathbf{s})| &\leq \sum_{N_0 \leq n} \sum_m p^{-\frac{n}{e}(e\Re(s_1) + |d|\Re(s_2))} \cdot p^{\frac{(n|d|-me)}{e}\Re(s_2)} \cdot |\eta_{\alpha,n,m}(\mathbf{s})| \\
&\leq \sum_{N_0 \leq n} p^{-\frac{n}{e}(e\Re(s_1) + |d|\Re(s_2))} \cdot \frac{p^{\frac{c}{e}\Re(s_2)}}{1 - p^{-\frac{\Re(s_2)}{e}}}
\end{aligned}$$

Thus $\eta_\alpha(\mathbf{s})$ is holomorphic on T_Λ.

□

From the proof of Proposition 3.7, we can claim more for finite places:

Proposition 3.8. *Let $\epsilon > 0$ be any small positive real number. Fix a compact subset K of Λ, and assume that $\Re(\mathbf{s})$ is in K. Define:*

$$\kappa(K) := \begin{cases} \max\left\{0, -\frac{\Re(s_1)}{|d|}, -\frac{\Re(s_2)}{|e|}\right\} & \text{if } d < 0 \text{ and } e < 0, \\ \max\left\{0, -\frac{\Re(s_1)}{|d|}\right\} & \text{if } d < 0 \text{ and } e \geq 0,. \end{cases}$$

Then we have

$$|\eta_\alpha(\mathbf{s})| \ll 1/|\alpha|_p^{\kappa(K)+\epsilon}$$

as $|\alpha|_p \to 0$.

Proof. Let $|\alpha|_p = p^{-k}$, and assume that both d, e are negative. By changing variables, if necessary, we may assume that N_0 in the proof of Proposition 3.7 is zero. If k is sufficiently large, then one can prove that there exists a constant c such that $\eta_{\alpha,n,m}(\mathbf{s}) = 0$ if $n|d| + m|e| > k + c$. Also it is easy to see that

$$|p^{-(ns_1+ms_2)}| \leq p^{(n|d|+m|e|)\kappa(K)}.$$

Hence we can conclude that

$$|\eta_\alpha(\mathbf{s})| \ll k^2 1/|\alpha|_p^{\kappa(K)} \ll 1/|\alpha|_p^{\kappa(K)+\epsilon}.$$

The case of $d<0$ and $e=0$ is treated similarly.

Assume that $d<0$ and $e>0$. Then we have a constant c such that $\eta_{\alpha,n,m}(\mathbf{s})=0$ if $n|d|-me>k+c$. Thus we can conclude that

$$\begin{aligned}
|\eta_\alpha(\mathbf{s})| &\le \sum_{m\ge 0}\sum_{n\ge 0} p^{-(n\Re(s_1)+m\Re(s_2))}|\eta_{\alpha,n,m}(\mathbf{s})| \\
&\ll \sum_{m\ge 0} p^{-m\Re(s_2)}(me+k)p^{(me+k)\kappa(K,s_2)} \\
&\ll k1/|\alpha|_p^{\kappa(K,s_2)}\sum_{m\ge 0}(m+1)p^{-m(\Re(s_2)-e\kappa(K,s_2))} \\
&\ll 1/|\alpha|_p^{\kappa(K,s_2)+\epsilon}.
\end{aligned}$$

where

$$\kappa(K,s_2)=\max\left\{0,-\frac{\Re(s_1)}{|d|} : (\Re(s_1),\Re(s_2))\in K\right\}.$$

Thus we can conclude that

$$|\eta_\alpha(\mathbf{s})| \ll 1/|\alpha|_p^{\kappa(K,s_2)+\epsilon} \ll 1/|\alpha|_p^{\kappa(K)+\epsilon}.$$

□

4. The projective plane

In this section, we implement the program described in Section 2 for the simplest equivariant compactifications of $G=G_1=\mathbb{G}_a\rtimes\mathbb{G}_m$, namely, the projective plane $\mathbb{P}^2$, for a *one-sided*, right, action of G given by

$$G\ni(x,a)\mapsto[x_0:x_1:x_2]=(a:a^{-1}x:1)\in\mathbb{P}^2.$$

The boundary consists of two lines, D_0 and D_2 given by the vanishing of x_0 and x_2. We will use the following identities:

$$\begin{aligned}
\mathrm{div}(a)&=D_0-D_2,\\
\mathrm{div}(x)&=D_0+D_1-2D_2,\\
\mathrm{div}(\omega)&=-3D_2,
\end{aligned}$$

where D_1 is given by the vanishing of x_1 and ω is the right invariant top degree form. The height functions are given by

$$\begin{aligned}
\mathsf{H}_{D_0,p}(a,x)&=\frac{\max\{|a|_p,|a^{-1}x|_p,1\}}{|a|_p}, &\quad \mathsf{H}_{D_2,p}(a,x)&=\max\{|a|_p,|a^{-1}x|_p,1\},\\
\mathsf{H}_{D_0,\infty}(a,x)&=\frac{\sqrt{|a|^2+|a^{-1}x|^2+1}}{|a|}, &\quad \mathsf{H}_{D_2,p}(a,x)&=\sqrt{|a|^2+|a^{-1}x|^2+1},\\
\mathsf{H}_{D_0}&=\prod_p \mathsf{H}_{D_0,p}\times\mathsf{H}_{D_0,\infty}, &\quad \mathsf{H}_{D_2}&=\prod_p \mathsf{H}_{D_2,p}\times\mathsf{H}_{D_2,\infty},
\end{aligned}$$

and the height pairing by

$$\mathsf{H}(\mathbf{s},g)=\mathsf{H}_{D_0}^{s_0}(g)\mathsf{H}_{D_2}^{s_2}(g),$$

for $\mathbf{s} = s_0 D_0 + s_2 D_2$ and $g \in G(\mathbb{A})$. The height zeta function takes the form

$$\mathsf{Z}(\mathbf{s}, g) = \sum_{\gamma \in G(\mathbb{Q})} \mathsf{H}(\mathbf{s}, \gamma g)^{-1}.$$

The proof of Northcott's theorem shows that the Dirichlet series $\mathsf{Z}(\mathbf{s}, g)$ converges absolutely and normally to a holomorphic function, for $\Re(\mathbf{s})$ is sufficiently large, which is continuous in $g \in G(\mathbb{A})$. Moreover, if $\Re(\mathbf{s})$ is sufficiently large, then

$$\mathsf{Z}(\mathbf{s}, g) \in \mathsf{L}^2(G(\mathbb{Q})\backslash G(\mathbb{A})) \cap \mathsf{L}^1(G(\mathbb{Q})\backslash G(\mathbb{A})).$$

According to Proposition 3.1, we have the following decomposition:

$$\mathsf{L}^2(G(\mathbb{Q})\backslash G(\mathbb{A})) = \mathsf{L}^2(\mathbb{G}_m(\mathbb{Q})\backslash \mathbb{G}_m(\mathbb{A})) \oplus \pi,$$

and we can write

$$\mathsf{Z}(\mathbf{s}, g) = \mathsf{Z}_0(\mathbf{s}, g) + \mathsf{Z}_1(\mathbf{s}, g).$$

The analysis of $\mathsf{Z}_0(\mathbf{s}, \mathrm{id})$ is a special case of our considerations in Section 2, in particular Theorem 2.1 (for further details, see [BT98] and [CLT10]). The conclusion here is that there exist a $\delta > 0$ and a function h which is holomorphic on the tube domain $\mathsf{T}_{>3-\delta}$ such that

$$\mathsf{Z}_0(\mathbf{s}, \mathrm{id}) = \frac{\mathsf{h}(s_0 + s_2)}{(s_0 + s_2 - 3)}.$$

The analysis of $\mathsf{Z}_1(\mathbf{s}, \mathrm{id})$, i.e., of the contribution from the unique infinite-dimensional representation occurring in $\mathsf{L}^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$, is the main part of this section. Define

$$\mathbf{K} = \prod_p \mathbf{K}_p \cdot \mathbf{K}_\infty = \prod_p G(\mathbb{Z}_p) \cdot \{(0, \pm 1)\}.$$

Since the height functions are $\mathbf{K}$-invariant,

$$\mathsf{Z}_1(\mathbf{s}, g) \in \pi^{\mathbf{K}} \simeq \mathsf{L}^2(\mathbb{A}^\times)^{\mathbf{K}}.$$

Lemma 3.5 provides a choice of an orthonormal basis for $\mathsf{L}^2(\mathbb{A}_{\mathrm{fin}}^\times)$. Combining with the Fourier expansion at the archimedean place, we obtain the following spectral expansion of Z_1:

Lemma 4.1. *Assume that $\Re(\mathbf{s})$ is sufficiently large. Then*

$$\mathsf{Z}_1(\mathbf{s}, g) = \sum_{m \geq 1} \frac{1}{4\pi} \int_{-\infty}^{\infty} (\mathsf{Z}(\mathbf{s}, g), \theta_{m,t}(g)) \theta_{m,t}(g) \mathrm{d}t,$$

where $\theta_{m,t}(g) = \Theta(\mathbf{v}_m \otimes |\cdot|^{it})(g)$.

Proof. See Lemma 5.3. □

It is easy to see that

$$\begin{aligned}
(\mathsf{Z}(\mathbf{s}, g), \theta_{m,t}(g)) &= \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \mathsf{Z}(\mathbf{s}, g) \theta_{m,t}(g) \mathrm{d}g \\
&= \int_{G(\mathbb{A})} \mathsf{H}(\boldsymbol{s}, g)^{-1} \bar{\theta}_{m,t}(g) \mathrm{d}g \\
&= \sum_{\alpha \in \mathbb{Q}^\times} \int_{G(\mathbb{A})} \mathsf{H}(\mathbf{s}, g)^{-1} \bar{\psi}(\alpha x) \mathbf{v}_m(\alpha a_{\mathrm{fin}}) |\alpha a_\infty|^{-it} \mathrm{d}g \\
&= \sum_{\alpha \in \mathbb{Q}^\times} \prod_p \mathsf{H}'_p(\mathbf{s}, m, \alpha) \cdot \mathsf{H}'_\infty(\mathbf{s}, t, \alpha),
\end{aligned}$$

where

$$\mathsf{H}'_p(\mathbf{s}, m, \alpha) = \int_{G(\mathbb{Q}_p)} \mathsf{H}_p(\mathbf{s}, g_p)^{-1} \bar{\psi}_p(\alpha x_p) \mathbf{1}_{m\mathbb{Z}_p^\times}(\alpha a_p) \mathrm{d}g_p,$$
$$\mathsf{H}'_\infty(\mathbf{s}, t, \alpha) = \int_{G(\mathbb{R})} \mathsf{H}_\infty(\mathbf{s}, g_\infty)^{-1} \bar{\psi}_\infty(\alpha x_\infty) |\alpha a_\infty|^{-it} \mathrm{d}g_\infty.$$

Note that $\theta_{m,t}(\mathrm{id}) = 2|m|^{it}$. Hence we can conclude that

$$\mathsf{Z}_1(\mathbf{s}, \mathrm{id}) = \sum_{\alpha \in \mathbb{Q}^\times} \sum_{m=1}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \prod_p \mathsf{H}'_p(\mathbf{s}, m, \alpha) \cdot \mathsf{H}'_\infty(\mathbf{s}, t, \alpha) |m|^{it} \mathrm{d}t$$

$$= \sum_{\alpha \in \mathbb{Q}^\times} \sum_{m=1}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \prod_p \int_{G(\mathbb{Q}_p)} \mathsf{H}_p(\mathbf{s}, g_p)^{-1} \bar{\psi}_p(\alpha x_p) \mathbf{1}_{m\mathbb{Z}_p^\times}(\alpha a_p) |\alpha a|_p^{-it} \mathrm{d}g_p \cdot \mathsf{H}'_\infty(\mathbf{s}, t, \alpha) \mathrm{d}t$$

$$= \sum_{\alpha \in \mathbb{Q}^\times} \frac{1}{2\pi} \int_{-\infty}^{\infty} \prod_p \widehat{\mathsf{H}}_p(\mathbf{s}, \alpha, t) \cdot \widehat{\mathsf{H}}_\infty(\mathbf{s}, \alpha, t) \mathrm{d}t,$$

where

$$\widehat{\mathsf{H}}_p(\mathbf{s}, \alpha, t) = \int_{G(\mathbb{Q}_p)} \mathsf{H}_p(\mathbf{s}, g_p)^{-1} \bar{\psi}_p(\alpha x_p) \mathbf{1}_{\mathbb{Z}_p}(\alpha a_p) |a_p|_p^{-it} \mathrm{d}g_p,$$
$$\widehat{\mathsf{H}}_\infty(\mathbf{s}, \alpha, t) = \int_{G(\mathbb{R})} \mathsf{H}_\infty(\mathbf{s}, g_\infty)^{-1} \bar{\psi}_\infty(\alpha x_\infty) |a_\infty|^{-it} \mathrm{d}g_\infty.$$

It is clear that

$$\widehat{\mathsf{H}}_p(\mathbf{s}, \alpha, t) = \widehat{\mathsf{H}}_p((s_0 - it, s_2 + it), \alpha, 0),$$

so we only need to study $\widehat{\mathsf{H}}_p(\mathbf{s}, \alpha) = \widehat{\mathsf{H}}_p(\mathbf{s}, \alpha, 0)$. To do this, we introduce some notation. We have the canonical integral model of $\mathbb{P}^2$ over $\mathrm{Spec}(\mathbb{Z})$, and for any prime p, we have the reduction map modulo p:

$$\rho : G(\mathbb{Q}_p) \subset \mathbb{P}^2(\mathbb{Q}_p) = \mathbb{P}^2(\mathbb{Z}_p) \to \mathbb{P}^2(\mathbb{F}_p)$$

This is a continuous map from $G(\mathbb{Q}_p)$ to $\mathbb{P}^2(\mathbb{F}_p)$. Consider the following open sets:

$$U_\phi = \rho^{-1}(\mathbb{P}^2 \setminus (D_0 \cup D_2)) = \{|a|_p = 1, |x|_p \leq 1\}$$
$$U_{D_0} = \rho^{-1}(D_0 \setminus (D_0 \cap D_2)) = \{|a|_p < 1, |a^{-1}x|_p \leq 1\}$$
$$U_{D_2} = \rho^{-1}(D_2 \setminus (D_0 \cap D_2)) = \{|a|_p > 1, |a^{-2}x|_p \leq 1\}$$
$$U_{D_0, D_2} = \rho^{-1}(D_0 \cap D_2) = \{|a^{-1}x|_p > 1, |a^{-2}x|_p > 1\}.$$

The height functions have a partial left invariance, i.e., they are invariant under the left action of the compact subgroup $\{(0, b) \mid b \in \mathbb{Z}_p^\times\}$. This implies that

$$\widehat{\mathsf{H}}_p(\mathbf{s}, \alpha) = \int_{G(\mathbb{Q}_p)} \mathsf{H}_p(\mathbf{s}, g)^{-1} \int_{\mathbb{Z}_p^\times} \bar{\psi}_p(\alpha b x) \mathrm{d}b^\times \mathbf{1}_{\mathbb{Z}_p}(\alpha a) \mathrm{d}g.$$

We record the following useful lemma (see, e.g., [CLT02, Lemma 10.3]):

Lemma 4.2.

$$\int_{\mathbb{Z}_p^\times} \bar{\psi}_p(bx)\mathrm{d}b^\times = \begin{cases} 1 & \text{if } |x|_p \leq 1, \\ -\frac{1}{p-1} & \text{if } |x|_p = p, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.3. *Assume that* $|\alpha|_p = 1$. *Then*

$$\widehat{\mathsf{H}}_p(\mathbf{s}, \alpha) = \frac{\zeta_p(s_0+1)\zeta_p(2s_0+s_2)}{\zeta_p(s_0+s_2)}.$$

Proof. We apply Lemma 4.2 and obtain

$$\begin{aligned}
\widehat{\mathsf{H}}_p(\boldsymbol{s}, \alpha) &= \int_{U_\emptyset} + \int_{U_{D_0}} + \int_{U_{D_0,D_2}} \\
&= 1 + \frac{p^{-(s_0+1)}}{1-p^{-(s_0+1)}} + \frac{p^{-(2s_0+s_2)} - p^{-(s_0+s_2)}}{(1-p^{-(s_0+1)})(1-p^{-(2s_0+s_2)})} \\
&= \frac{\zeta_p(s_0+1)\zeta_p(2s_0+s_2)}{\zeta_p(s_0+s_2)}.
\end{aligned}$$

□

Lemma 4.4. *Assume that* $|\alpha|_p > 1$. *Let* $|\alpha|_p = p^k$. *Then*

$$\widehat{\mathsf{H}}_p(\mathbf{s}, \alpha) = p^{-k(s_0+1)}\widehat{\mathsf{H}}_p(\mathbf{s}, 1).$$

Proof. Use the lemma, and we obtain that

$$\begin{aligned}
\widehat{\mathsf{H}}_p(\mathbf{s}, \alpha) &= \int_{U_{D_0}} + \int_{U_{D_0,D_2}} \\
&= \frac{p^{-k(s_0+1)}}{1-p^{-(s_0+1)}} + p^{-k(s_0+1)}\frac{p^{-(2s_0+s_2)} - p^{-(s_0+s_2)}}{(1-p^{-(s_0+1)})(1-p^{-(2s_0+s_2)})} \\
&= p^{-k(s_0+1)}\widehat{\mathsf{H}}_p(\mathbf{s}, 1).
\end{aligned}$$

□

Lemma 4.5. *Assume that* $|\alpha|_p < 1$. *Let* $|\alpha|_p = p^{-k}$. *Then* $\widehat{\mathsf{H}}_p(\mathbf{s}, \alpha)$ *is holomorphic on the tube domain* T_Λ *over the cone*

$$\Lambda = \{s_0 > -1, s_0 + s_2 > 0, 2s_0 + s_2 > 0\}.$$

Moreover, for any compact subset of Λ, *there exists a constant* $C > 0$ *such that*

$$|\widehat{\mathsf{H}}_p(\mathbf{s}, \alpha)| \leq Ck \max\{1, p^{-\frac{k}{2}\Re(s_2-2)}\}$$

for any $\mathbf{s}$ *with real part in this compact set.*

Proof. It is easy to see that

$$\begin{aligned}
\widehat{\mathsf{H}}_p(\mathbf{s}, \alpha) &= \int_{U_\emptyset} + \int_{U_{D_0}} + \int_{U_{D_2}} + \int_{U_{D_0,D_2}} \\
&= 1 + \frac{p^{-(s_0+1)}}{1-p^{-(s_0+1)}} + \int_{U_{D_2}} + \int_{U_{D_0,D_2}}.
\end{aligned}$$

On U_{D_2}, we choose $(1 : x_1 : x_2)$ as coordinates, then we have

$$\int_{U_{D_2}} = \int_{|x_1|_p \leq 1, |x_2|_p < 1} |x_2|_p^{s_2-2} \int_{\mathbb{Z}_p^\times} \bar{\psi}_p(\alpha b \frac{x_1}{x_2^2}) \mathrm{d}b^\times \mathbf{1}_{\mathbb{Z}_p}(\alpha x_2^{-1}) \mathrm{d}x_1 \, \mathrm{d}x_2^\times$$
$$= \int_{p^{-\frac{k}{2}} \leq |x_2|_p < 1} |x_2|_p^{s_2-2} \mathrm{d}x_2^\times$$

Hence this integral is holomorphic everywhere, and we have

$$\left| \int_{U_{D_2}} \right| < k \max\{1, p^{-\frac{k}{2}\mathrm{Re}(s_2-2)}\}.$$

On U_{D_0,D_2}, we choose $(x_0 : 1 : x_2)$ as coordinates and obtain

$$\frac{1}{1-p^{-1}} \int_{U_{D_0,D_2}} = \int_{|x_0|_p, |x_2|_p < 1} |x_0|_p^{s_0+1} |x_2|_p^{s_2-2} \int_{\mathbb{Z}_p^\times} \bar{\psi}_p(\alpha b \frac{x_0}{x_2^2}) \mathrm{d}b^\times \mathrm{d}x_0^\times \mathrm{d}x_2^\times$$
$$= -\frac{p^{-1}}{1-p^{-1}} p^{(k+1)(s_0+1)} \frac{p^{-(1-\left[-\frac{k}{2}\right])(2s_0+s_2)}}{1-p^{-(2s_0+s_2)}} + \int |x_0|_p^{s_0+1} |x_2|_p^{s_2-2} \mathrm{d}x_0^\times \mathrm{d}x_2^\times,$$

where the last integral is over

$$\{|x_0|_p < 1, |x_2|_p < 1, |\alpha x_0|_p \leq |x_2|_p^2\},$$

and is equal to

$$\sum_{j,l \geq 1, 2j-l \leq k} p^{-l(s_0+1)-j(s_2-2)}$$

$$= \sum_{j \leq \left[\frac{k}{2}\right]} p^{-j(s_2-2)} \frac{p^{-(s_0+1)}}{1-p^{-(s_0+1)}} + \sum_{j > \left[\frac{k}{2}\right]} p^{-j(s_2-2)} \frac{p^{-(2j-k)(s_0+1)}}{1-p^{-(s_{D_0}+1)}}$$

$$= \sum_{j \leq \left[\frac{k}{2}\right]} p^{-j(s_2-2)} \frac{p^{-(s_0+1)}}{1-p^{-(s_0+1)}} + \frac{p^{-(\left[\frac{k}{2}\right]+1)(2s_0+s_2)}}{1-p^{-(2s_0+s_2)}} \frac{p^{k(s_0+1)}}{1-p^{-(s_0+1)}}.$$

From this we can see that $\int_{U_{D_0,D_2}}$ is holomorphic on T_Λ and that for any compact subset of Λ, we can find a constant $C > 0$ such that

$$\left| \int_{U_{D_0,D_2}} \right| < Ck \max\{1, p^{-\frac{k}{2}\Re(s_2-2)}\},$$

for $\Re(\mathbf{s})$ in this compact. $\square$

Next, we study the local integral at the real place. Again,

$$\widehat{\mathsf{H}}_\infty(\mathbf{s}, \alpha, t) = \widehat{\mathsf{H}}_\infty((s_0 - it, s_2 + it), \alpha, 0),$$

and we start with

$$\widehat{\mathsf{H}}_\infty(\mathbf{s}, \alpha) = \widehat{\mathsf{H}}_\infty(\mathbf{s}, \alpha, 0).$$

Lemma 4.6. *The function*

$$\mathbf{s} \mapsto \widehat{\mathsf{H}}_\infty(\mathbf{s}, \alpha),$$

is holomorphic on the tube domain $\mathsf{T}_{\Lambda'}$ *over*

$$\Lambda' = \{s_0 > -1, s_2 > 0, 2s_0 + s_2 > 0\}.$$

Moreover, for any $r \in \mathbb{N}$ and any compact subset of

$$\Lambda'_r = \{s_0 > -1 + r, s_2 > 0\},$$

there exists a constant $C > 0$ such that

$$|\widehat{\mathsf{H}}_\infty(\mathbf{s}, \alpha)| < \frac{C}{|\alpha|_\infty^r},$$

for any $\mathbf{s}$ in the tube domain over this compact.

Proof. Let $U_\emptyset = X(\mathbb{R}) \setminus (D_0 \cup D_2)$, U_{D_i} be a small tubular neighborhood of D_i minus $D_0 \cap D_2$, and U_{D_0,D_2} be a small neighborhood of $D_0 \cap D_2$. Then $\{U_\emptyset, U_{D_0}, U_{D_2}, U_{D_0,D_2}\}$ is an open covering of $X(\mathbb{R})$, and consider the partition of unity for this covering; $\theta_\emptyset, \theta_{D_0}, \theta_{D_2}, \theta_{D_0,D_2}$. Then we have

$$\widehat{\mathsf{H}}_\infty(\mathbf{s}, \alpha) = \int_{U_\emptyset} \mathsf{H}_\infty^{-1} \bar{\psi}_\infty(\alpha x_\infty)\theta_\emptyset \mathrm{d}g_\infty + \int_{U_{D_0}} + \int_{U_{D_2}} + \int_{U_{D_0,D_2}}.$$

On U_{D_0,D_2}, we choose $(x_0 : 1 : x_2)$ as analytic coordinates and obtain

$$\int_{U_{D_0,D_2}} = \int_{\mathbb{R}^2} |x_0|^{s_0+1}|x_2|^{s_2-2}\bar{\psi}(\alpha\frac{x_0}{x_2^2})\Phi(\boldsymbol{s}, x_0, x_2)\mathrm{d}x_0^\times dx_2^\times,$$

where Φ is a smooth bounded function with compact support. Such oscillatory integrals have been studied in [CLT09], in our case the integral is holomorphic if $\mathrm{Re}(s_0) > -1$ and $\mathrm{Re}(2s_0 + s_2) > 0$. Assume that $\Re(\mathbf{s})$ is sufficiently large. Integration by parts implies that

$$\int_{U_{D_0,D_2}} = \frac{1}{\alpha^r}\int_{\mathbb{R}^2} |x_0|^{s_0+1-r}|x_2|^{s_2-2+2r}\bar{\psi}(\alpha\frac{x_0}{x_2^2})\Phi'(\boldsymbol{s}, x_0, x_2)dx_0^\times \mathrm{d}x_2^\times,$$

and this integral is holomorphic if $\Re(s_0) > -1 + r$ and $\Re(s_2) > 2 - 2r$. Thus, our second assertion follows. The other integrals are studied similarly. $\square$

Lemma 4.7. *For any compact set $K \subset \Lambda'_2$, there exists a constant $C > 0$ such that*

$$|\widehat{\mathsf{H}}_\infty(\mathbf{s}, \alpha, t)| < \frac{C}{|\alpha|^2(1 + t^2)},$$

for any $\mathbf{s} \in \mathsf{T}_K$.

Proof. Consider a left invariant differential operator $\partial_a = a\partial/\partial a$. Integrating by parts we obtain that

$$\widehat{\mathsf{H}}_\infty(\mathbf{s}, \alpha, t) = -\frac{1}{t^2}\int_{G(\mathbb{R})} \partial_a^2 \mathsf{H}_\infty(\mathbf{s}, g_\infty)^{-1}\bar{\psi}_\infty(\alpha x_\infty)|a_\infty|^{-it}\mathrm{d}g_\infty,$$

According to [CLT02],

$$\partial_a^2 \mathsf{H}_\infty(\mathbf{s}, g_\infty)^{-1} = \mathsf{H}_\infty(\mathbf{s}, g_\infty)^{-1} \times (\text{a bounded smooth function}),$$

so we can apply the discussion of the previous proposition. $\square$

Lemma 4.8. *The Euler product*

$$\prod_p \widehat{\mathsf{H}}_p(\mathbf{s}, \alpha, t) \cdot \widehat{\mathsf{H}}_\infty(\mathbf{s}, \alpha, t),$$

is holomorphic on the tube domain T_Ω over

$$\Omega = \{s_0 > 0, s_2 > 0, 2s_0 + s_2 > 1\}.$$

Moreover, let $\alpha = \frac{\beta}{\gamma}$*, where* $\gcd(\beta, \gamma) = 1$*. Then for any* $\epsilon > 0$ *and any compact set*

$$K \subset \Omega' = \{s_0 > 1, s_2 > 0, 2s_0 + s_2 > 1\},$$

there exists a constant $C > 0$ *such that*

$$|\prod_p \widehat{\mathsf{H}}_p(\mathbf{s}, \alpha, t) \cdot \widehat{\mathsf{H}}_\infty(\mathbf{s}, \alpha, t)| < C \cdot \frac{\max\{1, \sqrt{|\beta|}^{-\Re(s_2-2)}\}}{|\beta|^{2-\epsilon}|\gamma|^{\Re(s_0-1)}},$$

for all $\mathbf{s} \in \mathsf{T}_K$*.*

Theorem 4.9. *There exists* $\delta > 0$ *such that* $\mathsf{Z}_1(s_0 + s_2, \mathrm{id})$ *is holomorphic on* $\mathsf{T}_{>3-\delta}$*.*

Proof. Let $\delta > 0$ be a sufficiently small real number, and define

$$\Lambda = \{s_0 > 2 + \delta, s_2 > 1 - 2\delta\}.$$

It follows from the previous proposition that for any $\epsilon > 0$ and any compact set $K \subset \Lambda$, there exists a constant $C > 0$ such that

$$|\prod_p \widehat{\mathsf{H}}_p(\mathbf{s}, \alpha, t) \cdot \widehat{\mathsf{H}}_\infty(\mathbf{s}, \alpha, t)| < \frac{C}{(1+t^2)|\beta|^{\frac{3}{2}-\epsilon-\delta}|\gamma|^{1+\delta}}.$$

From this inequality, we can conclude that the integral

$$\int_{-\infty}^{\infty} \prod_p \widehat{\mathsf{H}}_p(\mathbf{s}, \alpha, t) \cdot \widehat{\mathsf{H}}_\infty(\mathbf{s}, \alpha, t) \mathrm{d}t,$$

converges uniformly and absolutely to a holomorphic function on T_K. Furthermore, we have

$$\left| \int_{-\infty}^{\infty} \prod_p \widehat{\mathsf{H}}_p(\mathbf{s}, \alpha, t) \cdot \widehat{\mathsf{H}}_\infty(\mathbf{s}, \alpha, t) \mathrm{d}t \right| < \frac{C'}{|b|^{\frac{3}{2}-\epsilon-\delta}|c|^{1+\delta}}.$$

For sufficiently small $\epsilon > 0$ and $\delta > 0$, the sum

$$\sum_{\alpha \in \mathbb{Q}^\times} \frac{1}{2\pi} \int_{-\infty}^{\infty} \prod_p \widehat{\mathsf{H}}_p(\mathbf{s}, \alpha, t) \cdot \widehat{\mathsf{H}}_\infty(\mathbf{s}, \alpha, t) \mathrm{d}t,$$

converges absolutely and uniformly to a function in $s_0 + s_2$. This concludes the proof of our theorem. □

5. Geometrization

In this section we geometrize the method described in Section 4. Our main theorem is:

Theorem 5.1. *Let* X *be a smooth projective equivariant compactification of* $G = G_1$ *over* $\mathbb{Q}$*, under the right action. Assume that the boundary divisor has strict normal crossings. Let* $a, x \in \mathbb{Q}(X)$ *be rational functions, where* (x, a) *are the standard coordinates on* $G \subset X$*. Let* E *be the Zariski closure of* $\{x = 0\} \subset G$*. Assume that:*

- *the union of the boundary and* E *is a divisor with strict normal crossings,*
- $\mathrm{div}(a)$ *is a reduced divisor, and*
- *for any pole* D_ι *of* a*, one has*

$$-\mathrm{ord}_{D_\iota}(x) > 1.$$

Then Manin's conjecture holds for X.

The remainder of this section is devoted to a proof of this fact. Blowing up the zero-dimensional subscheme

$$\mathrm{Supp}(\mathrm{div}_0(a)) \cap \mathrm{Supp}(\mathrm{div}_\infty(a)),$$

if necessary, we may assume that

$$\mathrm{Supp}(\mathrm{div}_0(a)) \cap \mathrm{Supp}(\mathrm{div}_\infty(a)) = \emptyset.$$

The height functions are invariant under the right action of some compact subgroup $\mathbf{K}_p \subset G(\mathbb{Z}_p)$. Moreover, we can assume that $\mathbf{K}_p = G(p^{n_p}\mathbb{Z}_p)$, for some $n_p \in \mathbb{Z}_{\geq 0}$. Let S be the set of bad places for X; note that $n_p = 0$ for all $p \notin S$. For simplicity, we assume that the height function at the infinite place is invariant under the action of $\mathbf{K}_\infty = \{(0, \pm 1)\}$.

Lemma 5.2. *We have*

$$\mathsf{Z}(\mathbf{s}, g) \in \mathsf{L}^2(G(\mathbb{Q})\backslash G(\mathbb{A}))^{\mathbf{K}} \cap \mathsf{L}^1(G(\mathbb{Q})\backslash G(\mathbb{A})).$$

Proof. First it is easy to see that

$$\begin{aligned}\int_{G(\mathbb{Q})\backslash G(\mathbb{A})} |\mathsf{Z}(\mathbf{s}, g)| \,\mathrm{d}g &\leq \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \sum_{\gamma \in G(\mathbb{Q})} |\mathsf{H}(\mathbf{s}, \gamma g)|^{-1} \,\mathrm{d}g \\ &= \int_{G(\mathbb{A})} \mathsf{H}(\Re(\mathbf{s}), g)^{-1} \,\mathrm{d}g,\end{aligned}$$

and the last integral is bounded when $\Re(\mathbf{s})$ is sufficiently large. (See [CLT10, Proposition 4.3.4].) Hence it follows that $\mathsf{Z}(\mathbf{s}, g)$ is integrable. To conclude that $\mathsf{Z}(\mathbf{s}, g)$ is square-integrable, we prove that $\mathsf{Z}(\mathbf{s}, g) \in \mathsf{L}^\infty$ for $\Re(\mathbf{s})$ sufficiently large. Let u, v be sufficiently large positive real numbers. Assume that $\Re(\mathbf{s})$ is in a fixed compact subset of $\mathrm{Pic}^G(X) \otimes \mathbb{R}$ and sufficiently large. Then we have

$$\mathsf{H}(\Re(\mathbf{s}), g)^{-1} \ll \mathsf{H}_1(a)^{-u} \cdot \mathsf{H}_2(x)^{-v}$$

where

$$\begin{aligned}\mathsf{H}_1(a) &= \prod_p \max\{|a_p|_p, |a_p|_p^{-1}\} \cdot \sqrt{|a_\infty|_\infty^2 + |a_\infty|_\infty^{-2}} \\ \mathsf{H}_2(x) &= \prod_p \max\{1, |x_p|_p\} \cdot \sqrt{1 + |x_\infty|_\infty^2}.\end{aligned}$$

Since $\mathsf{Z}(\mathbf{s}, g)$ is $G(\mathbb{Q})$-periodic, we may assume that $|a_p|_p = 1$ where $g_p = (x_p, a_p)$. Then we obtain that

$$\begin{aligned}\sum_{\gamma \in G(\mathbb{Q})} \mathsf{H}(\Re(\mathbf{s}), \gamma g)^{-1} &\ll \sum_{\alpha \in \mathbb{Q}^\times} \sum_{\beta \in \mathbb{Q}} \mathsf{H}_1(\alpha a)^{-u} \cdot \mathsf{H}_2(\alpha x + \beta)^{-v} \\ &\leq \sum_{\alpha \in \mathbb{Q}^\times} \mathsf{H}_{1,\mathrm{fin}}(\alpha)^{-u} \mathsf{Z}_2(\alpha x),\end{aligned}$$

where

$$\mathsf{Z}_2(x) = \sum_{\beta \in \mathbb{Q}} \mathsf{H}_2(x + \beta)^{-v}.$$

It is known that Z_2 is a bounded function for sufficiently large v, (see [CLT02]) so we can conclude that $\mathsf{Z}(\mathbf{s},g)$ is also a bounded function because

$$\sum_{\alpha\in\mathbb{Q}^\times}\mathsf{H}_{1,\mathrm{fin}}(\alpha)^{-u}<+\infty,$$

for sufficiently large u. ☐

By Proposition 3.1, the height zeta function decomposes as

$$\mathsf{Z}(\mathbf{s},\mathrm{id})=\mathsf{Z}_0(\mathbf{s},\mathrm{id})+\mathsf{Z}_1(\mathbf{s},\mathrm{id}).$$

Analytic properties of $\mathsf{Z}_0(\mathbf{s},\mathrm{id})$ were established in Section 2. It remains to show that $\mathsf{Z}_1(\mathbf{s},\mathrm{id})$ is holomorphic on a tube domain over an open neighborhood of the shifted effective cone $-K_X+\Lambda_{\mathrm{eff}}(X)$. To conclude this, we use the spectral decomposition of Z_1:

Lemma 5.3. *We have*

$$\mathsf{Z}_1(\mathbf{s},\mathrm{id})=\sum_{\lambda\in\mathbf{M}}\sum_{m=1}^{\infty}\frac{1}{4\pi}\int_{-\infty}^{+\infty}(\mathsf{Z}(\mathbf{s},g),\theta_{m,\lambda,t})\theta_{m,\lambda,t}(\mathrm{id})\,\mathrm{d}t.$$

Proof. To apply Proposition 3.6, we need to check that Z_1 satisfies the assumptions of Proposition 3.6. The proof of Lemma 3.4 implies that

$$I(\mathsf{Z}_1)=\int_{N(\mathbb{Q})\backslash N(\mathbb{A})}\mathsf{Z}(\mathbf{s},g)\psi(x)\,\mathrm{d}x.$$

Thus we have

$$\begin{aligned}\int_{T(\mathbb{A})}|I(\mathsf{Z}_1)|\,\mathrm{d}a^\times&=\int_{T(\mathbb{A})}\left|\int_{N(\mathbb{Q})\backslash N(\mathbb{A})}\mathsf{Z}(\mathbf{s},g)\psi(x)\,\mathrm{d}x\right|\,\mathrm{d}a^\times\\&\le\sum_{\alpha\in\mathbb{Q}^\times}\int_{T(\mathbb{A})}\left|\int_{N(\mathbb{A})}\mathsf{H}(\mathbf{s},g)^{-1}\psi(\alpha x)\,\mathrm{d}x\right|\,\mathrm{d}a^\times\\&=\sum_{\alpha\in\mathbb{Q}^\times}\prod_p\int_{T(\mathbb{Q}_p)}\left|\int_{N(\mathbb{Q}_p)}\mathsf{H}_p(\mathbf{s},g_p)^{-1}\psi_p(\alpha x_p)\,\mathrm{d}x_p\right|\mathrm{d}a_p^\times\\&\qquad\qquad\times\int_{T(\mathbb{R})}\left|\int_{N(\mathbb{R})}\mathsf{H}_\infty(\mathbf{s},g_\infty)^{-1}\psi_\infty(\alpha x)\,\mathrm{d}x_\infty\right|\,\mathrm{d}a_\infty^\times.\end{aligned}$$

Assume that $p\notin S$. Since the height function is right $\mathbf{K}_p$-invariant, we obtain that for any $y_p\in\mathbb{Z}_p$,

$$\begin{aligned}\int_{N(\mathbb{Q}_p)}\mathsf{H}_p(\mathbf{s},g_p)^{-1}\psi_p(\alpha x_p)\,\mathrm{d}x_p&=\int_{N(\mathbb{Q}_p)}\mathsf{H}_p(\mathbf{s},(x_p+a_py_p,a_p))^{-1}\psi_p(\alpha x_p)\,\mathrm{d}x_p\\&=\int_{N(\mathbb{Q}_p)}\mathsf{H}_p(\mathbf{s},g_p)^{-1}\psi_p(\alpha x_p)\int_{\mathbb{Z}_p}\overline{\psi}_p(\alpha a_py_p)\,\mathrm{d}y_p\,\mathrm{d}x_p\\&=0\qquad\text{if }|\alpha a_p|_p>1.\end{aligned}$$

Hence we can conclude that

$$\int_{T(\mathbb{Q}_p)}\left|\int_{N(\mathbb{Q}_p)}\mathsf{H}_p(\mathbf{s},g_p)^{-1}\psi_p(\alpha x_p)\,\mathrm{d}x_p\right|\mathrm{d}a_p^\times\le\int_{G(\mathbb{Q}_p)}\mathsf{H}_p(\Re(\mathbf{s}),g_p)^{-1}\mathbf{1}_{\mathbb{Z}_p}(\alpha a_p)\,\mathrm{d}g_p.$$

Similarly, for $p \in S$, we can conclude that

$$\int_{T(\mathbb{Q}_p)} \left| \int_{N(\mathbb{Q}_p)} \mathsf{H}_p(\mathbf{s}, g_p)^{-1} \psi_p(\alpha x_p) \, \mathrm{d}x_p \right| \mathrm{d}a_p^\times \leq \int_{G(\mathbb{Q}_p)} \mathsf{H}_p(\Re(\mathbf{s}), g_p)^{-1} \mathbf{1}_{\frac{1}{N}\mathbb{Z}_p}(\alpha a_p) \, \mathrm{d}g_p.$$

Then the convergence of the following sum

$$\sum_{\alpha \in \mathbb{Q}^\times} \prod_p \int_{G(\mathbb{Q}_p)} \mathsf{H}_p^{-1} \mathbf{1}_{\frac{1}{N}\mathbb{Z}_p}(\alpha a_p) \, \mathrm{d}g_p \cdot \int_{T(\mathbb{R})} \left| \int_{N(\mathbb{R})} \mathsf{H}_\infty^{-1} \psi_\infty(\alpha x) \, \mathrm{d}x_\infty \right| \mathrm{d}a_\infty^\times,$$

can be verified from the detailed study of the local integrals which we will conduct later. See proofs of Lemmas 5.6, 5.9, and 5.10.

Next we need to check that

$$\int_{-\infty}^{+\infty} |(\mathsf{Z}(\mathbf{s}, g), \theta_{m,\lambda,t})| \, \mathrm{d}t < +\infty.$$

It is easy to see that

$$\begin{aligned}
(\mathsf{Z}(\mathbf{s}, g), \theta_{m,\lambda,t}) &= \int_{G(\mathbb{Q}) \backslash G(\mathbb{A}_\mathbb{Q})} \mathsf{Z}(\mathbf{s}, g) \overline{\theta}_{m,\lambda,t} \, \mathrm{d}g \\
&= \int_{G(\mathbb{A}_\mathbb{Q})} \mathsf{H}(\mathbf{s}, g)^{-1} \overline{\theta}_{m,\lambda,t} \, \mathrm{d}g \\
&= \sum_{\alpha \in \mathbb{Q}^\times} \int_{G(\mathbb{A}_\mathbb{Q})} \mathsf{H}(\mathbf{s}, g)^{-1} \overline{\psi}(\alpha x) \overline{\mathbf{v}}_{m,\lambda}(\alpha a_{\mathrm{fin}}) |\alpha a_\infty|_\infty^{-it} \, \mathrm{d}g \\
&= \sum_{\alpha \in \mathbb{Q}^\times} \prod_p \mathsf{H}'_p(\mathbf{s}, m, \lambda, \alpha) \cdot \mathsf{H}'_\infty(\mathbf{s}, t, \alpha),
\end{aligned}$$

where $\mathsf{H}'_p(\mathbf{s}, m, \lambda, \alpha)$ is given by

$$\begin{aligned}
&= \int_{G(\mathbb{Q}_p)} \mathsf{H}_p(\mathbf{s}, g_p)^{-1} \overline{\psi}_p(\alpha x_p) \mathbf{1}_{m\mathbb{Z}_p^\times}(\alpha a_p) \, \mathrm{d}g_p, && p \notin S \\
&= \int_{G(\mathbb{Q}_p)} \mathsf{H}_p(\mathbf{s}, g_p)^{-1} \overline{\psi}_p(\alpha x_p) \overline{\lambda}_p(\alpha a_p / p^{v_p(\alpha a_p)}) \mathbf{1}_{\frac{m}{N}\mathbb{Z}_p^\times}(\alpha a_p) \, \mathrm{d}g_p, && p \in S
\end{aligned}$$

and

$$\mathsf{H}'_\infty(\mathbf{s}, t, \alpha) = \int_{G(\mathbb{R})} \mathsf{H}_\infty(\mathbf{s}, g_\infty)^{-1} \overline{\psi}_\infty(\alpha x_\infty) |\alpha a_\infty|_\infty^{-it} \mathrm{d}g_\infty.$$

The integrability follows from the proof of Lemma 5.9. Thus we can apply Proposition 3.6, and the identity in our statement follows from the continuity of $\mathsf{Z}(\mathbf{s}, g)$. $\square$

We obtained that

$$\mathsf{Z}_1(\mathbf{s}, \mathrm{id}) = \sum_{\lambda \in \mathbf{M}} \sum_{m=1}^{\infty} \frac{1}{4\pi} \int_{-\infty}^{+\infty} (\mathsf{Z}(\mathbf{s}, g), \theta_{m,\lambda,t}) \theta_{m,\lambda,t}(\mathrm{id}) \, \mathrm{d}t$$

$$= \sum_{\lambda \in \mathbf{M},\, \lambda(-1)=1} \sum_{m=1}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{+\infty} (\mathsf{Z}(\mathbf{s}, g), \theta_{m,\lambda,t}) \prod_{p \in S} \lambda_p \left(\frac{m}{N} \cdot p^{-v_p(m/N)} \right) \left| \frac{m}{N} \right|_\infty^{it} \mathrm{d}t.$$

We will use the following notation:

$$\lambda_S(\alpha a_p) := \prod_{q\in S} \overline{\lambda}_q(p^{v_p(\alpha a_p)}), \qquad p \notin S$$

$$\lambda_{S,p}(\alpha a_p) := \overline{\lambda}_p\left(\frac{\alpha a_p}{p^{v_p(\alpha a_p)}}\right) \prod_{q\in S\setminus p} \overline{\lambda}_q(p^{v_p(\alpha a_p)}), \qquad p \in S.$$

Proposition 5.4. *If $\Re(s)$ is sufficiently large, then*

$$\mathsf{Z}_1(\mathbf{s}, id) = \sum_{\lambda\in\mathbf{M},\, \lambda(-1)=1} \sum_{\alpha\in\mathbb{Q}^\times} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \prod_p \widehat{\mathsf{H}}_p(\mathbf{s}, \lambda, t, \alpha) \cdot \widehat{\mathsf{H}}_\infty(\mathbf{s}, t, \alpha)\, \mathrm{d}t,$$

where $\widehat{\mathsf{H}}_p(\mathbf{s}, \lambda, t, \alpha)$ is given by

$$\int_{G(\mathbb{Q}_p)} \mathsf{H}_p(\mathbf{s}, g_p)^{-1} \overline{\psi}_p(\alpha x_p) \lambda_S(\alpha a_p) \mathbf{1}_{\mathbb{Z}_p}(\alpha a_p) |a_p|_p^{-it} \mathrm{d}g_p, \qquad p \notin S$$

$$\int_{G(\mathbb{Q}_p)} \mathsf{H}_p(\mathbf{s}, g_p)^{-1} \overline{\psi}_p(\alpha x_p) \lambda_{S,p}(\alpha a_p) \mathbf{1}_{\frac{1}{N}\mathbb{Z}_p}(\alpha a_p) |a_p|_p^{-it} \mathrm{d}g_p, \qquad p \in S$$

and

$$\widehat{\mathsf{H}}_\infty(\mathbf{s}, t, \alpha) = \int_{G(\mathbb{R})} \mathsf{H}_\infty(\mathbf{s}, g_\infty)^{-1} \overline{\psi}_\infty(\alpha x_\infty) |a_\infty|_\infty^{-it}\, \mathrm{d}g_\infty$$

Proof. For simplicity, we assume that $S = \emptyset$. We have seen that

$$\mathsf{Z}_1(\mathbf{s}, \mathrm{id}) = \sum_{m=1}^{\infty} \sum_{\alpha\in\mathbb{Q}^\times} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \prod_p \mathsf{H}'_p(\mathbf{s}, m, \alpha) \cdot \mathsf{H}'_\infty(\mathbf{s}, t, \alpha) |m|_\infty^{it}\, \mathrm{d}t.$$

On the other hand, it is easy to see that

$$\widehat{\mathsf{H}}_p(\mathbf{s}, t, \alpha) = \sum_{j=0}^{\infty} \int_{G(\mathbb{Q}_p)} \mathsf{H}_p(\mathbf{s}, g_p)^{-1} \overline{\psi}_p(\alpha x_p) \mathbf{1}_{p^j\mathbb{Z}_p^\times}(\alpha a_p) \left|\frac{p^j}{\alpha}\right|_p^{-it} \mathrm{d}g_p.$$

Hence we have the formal identity:

$$\prod_p \widehat{\mathsf{H}}_p(\mathbf{s}, t, \alpha) \cdot \widehat{\mathsf{H}}_\infty(\mathbf{s}, t, \alpha) = \sum_{m=1}^{\infty} \prod_p \mathsf{H}'_p(\mathbf{s}, m, \alpha) \cdot \mathsf{H}'_\infty(\mathbf{s}, t, \alpha) |m|_\infty^{it},$$

and our assertion follows from this. To justify the above identity, we need to address convergence issues; this will be discussed below (see the proof of Lemma 5.6). $\square$

Thus we need to study the local integrals in Proposition 5.4. We introduce some notation:

$$\mathcal{I}_1 = \{\iota \in \mathcal{I} \mid D_\iota \subset \mathrm{Supp}(\mathrm{div}_0(a))\}$$
$$\mathcal{I}_2 = \{\iota \in \mathcal{I} \mid D_\iota \subset \mathrm{Supp}(\mathrm{div}_\infty(a))\}$$
$$\mathcal{I}_3 = \{\iota \in \mathcal{I} \mid D_\iota \not\subset \mathrm{Supp}(\mathrm{div}(a))\}.$$

Note that $\mathcal{I} = \mathcal{I}_1 \sqcup \mathcal{I}_2 \sqcup \mathcal{I}_3$ and $\mathcal{I}_1 \neq \emptyset$. Also $D_\iota \subset \mathrm{Supp}(\mathrm{div}_\infty(x))$ for any $\iota \in \mathcal{I}_3$ because

$$D = \cup_{\iota\in\mathcal{I}} D_\iota = \mathrm{Supp}(\mathrm{div}(a)) \cup \mathrm{Supp}(\mathrm{div}_\infty(x)).$$

Let

$$-\mathrm{div}(\omega) = \sum_{\iota\in\mathcal{I}} d_\iota D_\iota,$$

where $\omega = \mathrm{d}x\mathrm{d}a/a$ is the top degree right invariant form on G. Note that ω defines a measure $|\omega|$ on an analytic manifold $G(\mathbb{Q}_v)$, and for any finite place p,

$$|\omega| = \left(1 - \frac{1}{p}\right) \mathrm{d}g_p,$$

where $\mathrm{d}g_p$ is the standard Haar measure defined in Section 3.

Lemma 5.5. *Consider an open convex cone Ω in $\mathrm{Pic}^G(X)_{\mathbb{R}}$, defined by the following relations:*

$$\begin{cases} s_\iota - d_\iota + 1 > 0 & \text{if } \iota \in \mathcal{I}_1 \\ s_\iota - d_\iota + 1 + e_\iota > 0 & \text{if } \iota \in \mathcal{I}_2 \\ s_\iota - d_\iota + 1 > 0 & \text{if } \iota \in \mathcal{I}_3 \end{cases}$$

where $e_\iota = |\mathrm{ord}_{D_\iota}(x)|$. Then $\widehat{\mathsf{H}}_p(\mathbf{s}, \lambda, t, \alpha)$ and $\widehat{\mathsf{H}}_\infty(\mathbf{s}, t, \alpha)$ are holomorphic on T_Ω.

Proof. First we prove our assertion for $\widehat{\mathsf{H}}_\infty$. We can assume that

$$\widehat{\mathsf{H}}_v(\mathbf{s}, t) = \widehat{\mathsf{H}}_v(\mathbf{s} - it\mathbf{m}(a), 0),$$

where $\mathbf{m}(a) \in \mathfrak{X}^*(G) \subset \mathrm{Pic}^G(X)$ is the character associated to the rational function a (by choosing an appropriate height function). It suffices to discuss the case when $t = 0$. Choose a finite covering $\{U_\eta\}$ of $X(\mathbb{R})$ by open subsets and local coordinates y_η, z_η on U_η such that the union of the boundary divisor D and E is locally defined by $y_\eta = 0$ or $y_\eta \cdot z_\eta = 0$. Choose a partition of unity $\{\theta_\eta\}$; the local integral takes the form

$$\widehat{\mathsf{H}}_\infty(\mathbf{s}, \alpha) = \sum_\eta \int_{G(\mathbb{R})} \mathsf{H}_\infty(\mathbf{s}, g_\infty)^{-1} \overline{\psi}_\infty(\alpha x_\infty) \theta_\eta \, \mathrm{d}g_\infty.$$

Each integral is a oscillatory integral in the variables y_η, z_η. For example, assume that U_η meets $D_\iota, D_{\iota'}$, where $\iota, \iota' \in \mathcal{I}_2$. Then

$$\int_{G(\mathbb{R})} \mathsf{H}_\infty(\mathbf{s}, g_\infty)^{-1} \overline{\psi}_\infty(\alpha x_\infty) \theta_\eta \mathrm{d}g_\infty$$
$$= \int_{\mathbb{R}^2} |y_\eta|^{s_\iota - d_\iota} |z_\eta|^{s_{\iota'} - d_{\iota'}} \overline{\psi}_\infty \left(\frac{\alpha f}{y_\eta^{e_\iota} z_\eta^{e_{\iota'}}} \right) \Phi(\mathbf{s}, y_\eta, z_\eta) \mathrm{d}y_\eta \, \mathrm{d}z_\eta,$$

where Φ is a smooth function with compact support and f is a nonvanishing analytic function. Shrinking U_η and changing variables, if necessary, we may assume that f is a constant. Proposition 3.7 implies that this integral is holomorphic everywhere. The other integrals can be studied similarly.

Next we consider finite places. Let p be a prime of good reduction. Since

$$\mathrm{Supp}(\mathrm{div}_0(a)) \cap \mathrm{Supp}(\mathrm{div}_\infty(a)) = \emptyset,$$

the smooth function $\mathbf{1}_{\mathbb{Z}_p}(\alpha a_p)$ extends to a smooth function h on $X(\mathbb{Q}_p)$. Let

$$U = \{h = 1\}.$$

Then

$$\widehat{\mathsf{H}}_p(\mathbf{s}, \lambda, \alpha) = \int_U \mathsf{H}_p(\mathbf{s}, g_p)^{-1} \overline{\psi}_p(\alpha x_p) \lambda_S(\alpha a_p) \mathrm{d}g_p.$$

Now the proof of [CLT10, Lemma 4.4.1] implies that this is holomorphic on T_Ω because $U \cap (\cup_{\iota \in \mathcal{I}_2} D_\iota(\mathbb{Q}_p)) = \emptyset$. Places of bad reduction are treated similarly. $\square$

Lemma 5.6. *Let $|\alpha|_p = p^k > 1$. Then, for any compact set in Ω and for any $\delta > 0$, there exists a constant $C > 0$ such that*

$$|\widehat{\mathsf{H}}_p(\mathbf{s}, \lambda, t, \alpha)| < C|\alpha|_p^{-\min_{\iota\in\mathcal{I}_1}\{\Re(s_\iota)-d_\iota+1-\delta\}},$$

for $\Re(\mathbf{s})$ in that compact set.

Proof. First assume that p is a good reduction place. Let $\rho : \mathcal{X}(\mathbb{Z}_p) \to \mathcal{X}(\mathbb{F}_p)$ be the reduction map modulo p where $\mathcal{X}$ is a smooth integral model of X over $\mathrm{Spec}(\mathbb{Z}_p)$. Note that

$$\rho(\{|a|_p < 1\}) \subset \cup_{\iota\in\mathcal{I}_1}\mathcal{D}_\iota(\mathbb{F}_p),$$

where $\mathcal{D}_\iota$ is the Zariski closure of D_ι in $\mathcal{X}$. Thus $\widehat{\mathsf{H}}_p(\mathbf{s}, \lambda, \alpha)$ is given by

$$\widehat{\mathsf{H}}_p(\mathbf{s}, \lambda, \alpha) = \sum_{\tilde{x}\in\cup_{\iota\in\mathcal{I}_1}\mathcal{D}_\iota(\mathbb{F}_p)} \int_{\rho^{-1}(\tilde{x})} \mathsf{H}_p(\mathbf{s}, g_p)^{-1}\overline{\psi}_p(\alpha x_p)\lambda_S(\alpha a_p)\mathbf{1}_{\mathbb{Z}_p}(\alpha a_p)\mathrm{d}g_p.$$

Let $\tilde{x} \in \mathcal{D}_\iota(\mathbb{F}_p)$ for some $\iota \in \mathcal{I}_1$, but $\tilde{x} \notin \mathcal{D}_{\iota'}(\mathbb{F}_p)$ for any $\iota' \in \mathcal{I} \setminus \{\iota\}$. Since p is a good reduction place, we can find analytic coordinates y, z such that

$$\begin{aligned}
\left|\int_{\rho^{-1}(\tilde{x})}\right| &\le \int_{\rho^{-1}(\tilde{x})} \mathsf{H}_p(\Re(\mathbf{s}), g_p)^{-1}\mathbf{1}_{\mathbb{Z}_p}(\alpha a_p)\mathrm{d}g_p \\
&= \left(1-\frac{1}{p}\right)^{-1}\int_{\rho^{-1}(\tilde{x})} \mathsf{H}_p(\Re(\mathbf{s})-\mathbf{d}, g_p)^{-1}\mathbf{1}_{\mathbb{Z}_p}(\alpha a_p)\mathrm{d}\tau_{X,p} \\
&= \left(1-\frac{1}{p}\right)^{-1}\int_{\mathfrak{m}_p^2} |y|_p^{\Re(s_\iota)-d_\iota}\mathbf{1}_{\mathbb{Z}_p}(\alpha y)\mathrm{d}y_p\mathrm{d}z_p \\
&= \frac{1}{p}\cdot\frac{p^{-k(\Re(s_\iota)-d_\iota+1)}}{1-p^{-(\Re(s_\iota)-d_\iota+1)}},
\end{aligned}$$

where $\mathrm{d}\tau_{X,p}$ is the local Tamagawa measure (see [CLT10, Section 2] for the definition). For the construction of such local analytic coordinates, see [Wei82], [Den87], or [Sal98]. If $\tilde{x} \in \mathcal{D}_\iota(\mathbb{F}_p)\cap\mathcal{D}_{\iota'}(\mathbb{F}_p)$ for $\iota \in \mathcal{I}_1$, $\iota' \in \mathcal{I}_3$, then we can find local analytic coordinates y, z such that

$$\begin{aligned}
\left|\int_{\rho^{-1}(\tilde{x})}\right| &\le \left(1-\frac{1}{p}\right)\int_{\mathfrak{m}_p^2} |y|_p^{\Re(s_\iota)-d_\iota+1}|z|_p^{\Re(s_{\iota'})-d_{\iota'}+1}\mathbf{1}_{\mathbb{Z}_p}(\alpha y)\mathrm{d}y_p^\times\mathrm{d}z_p^\times \\
&= \left(1-\frac{1}{p}\right)\frac{p^{-k(\Re(s_\iota)-d_\iota+1)}}{1-p^{-(\Re(s_\iota)-d_\iota+1)}}\frac{p^{-(\Re(s_{\iota'})-d_{\iota'}+1)}}{1-p^{-(\Re(s_{\iota'})-d_{\iota'}+1)}}.
\end{aligned}$$

If $\tilde{x} \in \mathcal{D}_\iota(\mathbb{F}_p)\cap\mathcal{D}_{\iota'}(\mathbb{F}_p)$ for $\iota, \iota' \in \mathcal{I}_1$, $\iota \neq \iota'$, then we can find analytic coordinates x, y such that

$$\begin{aligned}
\left|\int_{\rho^{-1}(\tilde{x})}\right| &\le \left(1-\frac{1}{p}\right)\int_{\mathfrak{m}_p^2} |y|_p^{\Re(s_\iota)-d_\iota+1}|z|_p^{\Re(s_{\iota'})-d_{\iota'}+1}\mathbf{1}_{\mathbb{Z}_p}(\alpha yz)\,\mathrm{d}y_p^\times\mathrm{d}z_p^\times \\
&\le \left(1-\frac{1}{p}\right)\int_{\mathfrak{m}_p^2} |yz|_p^{\min\{\Re(s_\iota)-d_\iota+1,\,\Re(s_{\iota'})-d_{\iota'}+1\}}\mathbf{1}_{\mathbb{Z}_p}(\alpha yz)\,\mathrm{d}y_p^\times\mathrm{d}z_p^\times \\
&= \left(1-\frac{1}{p}\right)\left((k-1)\frac{p^{-kr}}{1-p^{-r}} + \frac{p^{-(k+1)r}}{(1-p^{-r})^2}\right),
\end{aligned}$$

where

$$r = \min\{\Re(s_\iota)-d_\iota+1,\, \Re(s_{\iota'})-d_{\iota'}+1\}.$$

It follows from these inequalities and Lemma 9.4 in [CLT02] that there exists a constant $C > 0$, independent of p, satisfying the inequality in the statement.

Next assume that p is a bad reduction place. Choose an open covering $\{U_\eta\}$ of $\cup_{\iota\in\mathcal{I}_1} D_\iota(\mathbb{Q}_p)$ such that

$$(\cup_\eta U_\eta) \cap (\cup_{\iota\in\mathcal{I}_2} D_\iota(\mathbb{Q}_p)) = \emptyset,$$

and each U_η has analytic coordinates y_η, z_η. Moreover, we can assume that the boundary divisor is defined by $y_\eta = 0$ or $y_\eta \cdot z_\eta = 0$ on U_η. Let V be the complement of $\cup_{\iota\in\mathcal{I}_1} D_\iota(\mathbb{Q}_p)$, and consider the partition of unity for $\{U_\eta, V\}$ which we denote by $\{\theta_\eta, \theta_V\}$. If k is sufficiently large, then

$$\{\mathbf{1}_{\frac{1}{N}\mathbb{Z}_p}(\alpha a) = 1\} \cap \mathrm{Supp}(\theta_V) = \emptyset.$$

Hence if k is sufficiently large, then

$$|\widehat{\mathsf{H}}_p(\mathbf{s}, \lambda, \alpha)| \leq \sum_\eta \int_{U_\eta} \mathsf{H}_p(\Re(\mathbf{s}), g_p)^{-1} \mathbf{1}_{\frac{1}{N}\mathbb{Z}_p}(\alpha a_p) \cdot \theta_\eta \, \mathrm{d}g_p.$$

When U_η meets only one component $D_\iota(\mathbb{Q}_p)$ for $\iota \in \mathcal{I}_1$, then

$$\int_{U_\eta} \leq \int_{\mathbb{Q}_p^2} |y_\eta|_p^{\Re(s_\iota)-d_\iota} \mathbf{1}_{c\mathbb{Z}_p}(\alpha y_\eta) \Phi(\mathbf{s}, y_\eta, z_\eta) \, \mathrm{d}y_{\eta,p} \mathrm{d}z_{\eta,p} \ll p^{-k(\Re(s_\iota)-d_\iota+1)},$$

as $k \to \infty$, where c is some rational number and Φ is a smooth function with compact support. Other integrals are treated similarly. □

We record the following useful lemma (see, e.g., [CLT09, Lemma 2.3.1]):

Lemma 5.7. *Let d be a positive integer and $a \in \mathbb{Q}_p$. If $|a|_p > p$ and $p \nmid d$, then*

$$\int_{\mathbb{Z}_p^\times} \overline{\psi}_p(ax^d) \, \mathrm{d}x_p^\times = 0.$$

Moreover, if $|a|_p = p$ and $d = 2$, then

$$\int_{\mathbb{Z}_p^\times} \overline{\psi}_p(ax^d) \mathrm{d}x_p^\times = \begin{cases} \frac{\sqrt{p}-1}{p-1} \textit{ or } \frac{i\sqrt{p}-1}{p-1} & \text{if } pa \text{ is a quadratic residue,} \\ \frac{-\sqrt{p}-1}{p-1} \textit{ or } \frac{-i\sqrt{p}-1}{p-1} & \text{if } pa \text{ is a quadratic non-residue.} \end{cases}$$

Lemma 5.8. *Let $|\alpha|_p = p^{-k} < 1$. Consider an open convex cone Ω_ϵ in $\mathrm{Pic}(X)_\mathbb{R}$, defined by the following relations:*

$$\begin{cases} s_\iota - d_\iota + 1 > 0 & \text{if } \iota \in \mathcal{I}_1 \\ s_\iota - d_\iota + 2 + \epsilon > 0 & \text{if } \iota \in \mathcal{I}_2 \\ s_\iota - d_\iota + 1 > 0 & \text{if } \iota \in \mathcal{I}_3 \end{cases}$$

where $0 < \epsilon < 1/3$. Then, for any compact set in Ω_ϵ, there exists a constant $C > 0$ such that

$$|\widehat{\mathsf{H}}_p(\mathbf{s}, \lambda, t, \alpha)| < C|\alpha|_p^{-\frac{2}{3}(1+2\epsilon)},$$

for $\Re(\mathbf{s})$ in that compact set.

Proof. First assume that p is a good reduction place and that $p \nmid e_\iota$, for any $\iota \in \mathcal{I}_2$. We have

$$\widehat{\mathsf{H}}_p(\mathbf{s}, \lambda, \alpha) = \sum_{\tilde{x}\in\mathcal{X}(\mathbb{F}_p)} \int_{\rho^{-1}(\tilde{x})} \mathsf{H}_p(\mathbf{s}, g_p)^{-1} \overline{\psi}_p(\alpha x_p) \lambda_S(\alpha a_p) \mathbf{1}_{\mathbb{Z}_p}(\alpha a_p) \, \mathrm{d}g_p.$$

A formula of J. Denef (see [Den87, Theorem 3.1] or [CLT10, Proposition 4.1.7]) and Lemma 9.4 in [CLT02] give us an uniform bound:

$$|\sum_{\tilde{x}\notin\cup_{\iota\in\mathcal{I}_2}\mathcal{D}_\iota(\mathbb{F}_p)}| \leq \sum_{\tilde{x}\notin\cup_{\iota\in\mathcal{I}_2}\mathcal{D}_\iota(\mathbb{F}_p)} \int_{\rho^{-1}(\tilde{x})} \mathsf{H}_p(\Re(\mathbf{s}), g_p)^{-1}\,\mathrm{d}g_p.$$

Hence we need to study

$$\sum_{\tilde{x}\in\cup_{\iota\in\mathcal{I}_2}\mathcal{D}_\iota(\mathbb{F}_p)} \int_{\rho^{-1}(\tilde{x})} \mathsf{H}_p(\mathbf{s}, g_p)^{-1}\overline{\psi}_p(\alpha x_p)\lambda_S(\alpha a_p)\mathbf{1}_{\mathbb{Z}_p}(\alpha a_p)\,\mathrm{d}g_p.$$

Let $\tilde{x}\in\mathcal{D}_\iota(\mathbb{F}_p)$ for some $\iota\in\mathcal{I}_2$, but $\tilde{x}\notin\mathcal{D}_{\iota'}(\mathbb{F}_p)\cup\mathcal{E}(\mathbb{F}_p)$ for any $\iota'\in\mathcal{I}\setminus\{\iota\}$, where $\mathcal{E}$ is the Zariski closure of E in $\mathcal{X}$. Then we can find local analytic coordinates y, z such that

$$\int_{\rho^{-1}(\tilde{x})} = \left(1-\frac{1}{p}\right)^{-1}\int_{\mathfrak{m}_p^2} |y|_p^{s_\iota-d_\iota}\overline{\psi}_p(\alpha f/y^{e_\iota})\lambda_S(\alpha y^{-1})\mathbf{1}_{\mathbb{Z}_p}(\alpha y^{-1})\,\mathrm{d}y_p\mathrm{d}z_p,$$

where $f\in\mathbb{Z}_p[[y,z]]$ such that $f(0)\in\mathbb{Z}_p^\times$. Since p does not divide e_ι, there exists $g\in\mathbb{Z}_p[[y,z]]$ such that $f=f(0)g^{e_\iota}$. After a change of variables, we can assume that $f=u\in\mathbb{Z}_p^\times$. Lemma 5.7 implies that

$$\begin{aligned}\int_{\rho^{-1}(\tilde{x})} &= \frac{1}{p}\int_{\mathfrak{m}_p} |y|_p^{s_\iota-d_\iota+1}\lambda_S(\alpha y^{-1})\int_{\mathbb{Z}_p^\times}\overline{\psi}_p(\alpha u b^{e_\iota}/y^{e_\iota})\mathrm{d}b_p^\times\mathbf{1}_{\mathbb{Z}_p}(\alpha y^{-1})\mathrm{d}y_p^\times\\ &= \frac{1}{p}\int_{p^{-(k+1)}\leq|y^{e_\iota}|_p} |y|_p^{s_\iota-d_\iota+1}\lambda_S(\alpha y^{-1})\int_{\mathbb{Z}_p^\times}\overline{\psi}_p(\alpha u b^{e_\iota}/y^{e_\iota})\,\mathrm{d}b_p^\times\mathrm{d}y_p^\times\end{aligned}$$

Thus it follows from the second assertion of Lemma 5.7 that

$$\begin{aligned}\left|\int_{\rho^{-1}(\tilde{x})}\right| &\leq \frac{1}{p}\int_{p^{-(k+1)}\leq|y^{e_\iota}|} |y|_p^{\Re(s_\iota)-d_\iota+1}\left|\int_{\mathbb{Z}_p^\times}\overline{\psi}_p(\alpha u b^{e_\iota}/y^{e_\iota})\mathrm{d}b_p^\times\right|\mathrm{d}y_p^\times\\ &\leq \frac{1}{p}kp^{\frac{k}{e_\iota}(1+\epsilon)}+\frac{1}{p}p^{\frac{k+1}{e_\iota}(1+\epsilon)}\times\begin{cases}1 & \text{if } e_\iota>2\\ \frac{1}{\sqrt{p}-1} & \text{if } e_\iota=2\end{cases}\\ &\ll \frac{1}{p}kp^{\frac{2}{3}k(1+\epsilon)}.\end{aligned}$$

If $\tilde{x}\in\mathcal{D}_\iota(\mathbb{F}_p)\cap\mathcal{E}(\mathbb{F}_p)$, for some $\iota\in\mathcal{I}_2$, then we have

$$\begin{aligned}\int_{\rho^{-1}(\tilde{x})} &= \left(1-\frac{1}{p}\right)^{-1}\int_{\mathfrak{m}_p^2} |y|_p^{s_\iota-d_\iota}\overline{\psi}_p(\alpha z/y^{e_\iota})\lambda_S(\alpha y^{-1})\mathbf{1}_{\mathbb{Z}_p}(\alpha y^{-1})\,\mathrm{d}y_p\mathrm{d}z_p\\ &= \int_{\mathfrak{m}_p} |y|_p^{s_\iota-d_\iota+1}\lambda_S(\alpha y^{-1})\mathbf{1}_{\mathbb{Z}_p}(\alpha y^{-1})\int_{\mathfrak{m}_p}\overline{\psi}_p(\alpha z/y^{e_\iota})\mathrm{d}z_p\mathrm{d}y_p^\times\\ &= \frac{1}{p}\int_{p^{-(k+1)}\leq|y|_p^{e_\iota}<1} |y|_p^{s_\iota-d_\iota+1}\lambda_S(\alpha y^{-1})\,\mathrm{d}y_p^\times.\end{aligned}$$

Hence we obtain that

$$\left|\int_{\rho^{-1}(\tilde{x})}\right| \leq \frac{1}{p}\int_{p^{-(k+1)}\leq|y|_p^{e_\iota}<1} |y|_p^{\Re(s_\iota)-d_\iota+1}\,\mathrm{d}y_p^\times \leq kp^{\frac{k}{e_\iota}(1+\epsilon)} < kp^{\frac{2}{3}k(1+\epsilon)}.$$

If $\tilde{x} \in \mathcal{D}_\iota(\mathbb{F}_p) \cap \mathcal{D}_{\iota'}(\mathbb{F}_p)$ for some $\iota \in \mathcal{I}_2$ and $\iota' \in \mathcal{I}_3$, then it follows from Lemma 5.7

$$\begin{aligned}\int_{\rho^{-1}(\tilde{x})} &= \left(1-\frac{1}{p}\right)^{-1} \int_{\mathfrak{m}_p^2} |y|_p^{s_\iota - d_\iota} |z|_p^{s_{\iota'} - d_{\iota'}} \overline{\psi}_p\left(\frac{\alpha u}{y^{e_\iota} z^{e_{\iota'}}}\right) \lambda_S(\alpha y^{-1}) \mathbf{1}_{\mathbb{Z}_p}(\alpha y^{-1})\, \mathrm{d}y_p \mathrm{d}z_p \\ &= \left(1-\frac{1}{p}\right)^{-1} \int |y|_p^{s_\iota - d_\iota} |z|_p^{s_{\iota'} - d_{\iota'}} \lambda_S(\alpha y^{-1}) \int_{\mathbb{Z}_p^\times} \overline{\psi}_p\left(\frac{\alpha u b^{e_\iota}}{y^{e_\iota} z^{e_{\iota'}}}\right) \mathrm{d}b_p^\times\, \mathrm{d}y_p \mathrm{d}z_p,\end{aligned}$$

where the last integral is over the domain

$$\{(y,z) \in \mathfrak{m}_p^2 : p^{-(k+1)} \leq |y^{e_\iota} z^{e_{\iota'}}|_p\}.$$

We conclude that

$$\begin{aligned}\left|\int_{\rho^{-1}(\tilde{x})}\right| &\leq \left(1-\frac{1}{p}\right)^{-1} \int_{p^{-(k+1)} \leq |y^{e_\iota} z^{e_{\iota'}}|_p} |y|_p^{\Re(s_\iota) - d_\iota} |z|_p^{\Re(s_{\iota'}) - d_{\iota'}}\, \mathrm{d}y_p \mathrm{d}z_p \\ &\leq \left(1-\frac{1}{p}\right)^{-1} \int_{p^{-k} \leq |y^{e_\iota}|_p < 1} |y|_p^{\Re(s_\iota) - d_\iota}\, \mathrm{d}y_p \int_{\mathfrak{m}_p} |z|_p^{\Re(s_{\iota'}) - d_{\iota'}}\, \mathrm{d}z_p \\ &\leq k p^{\frac{k}{e_\iota}(1+\epsilon)} \frac{p^{-(\Re(s_{\iota'}) - d_{\iota'} + 1)}}{1 - p^{-(\Re(s_{\iota'}) - d_{\iota'} + 1)}}.\end{aligned}$$

If $\tilde{x} \in \mathcal{D}_\iota(\mathbb{F}_p) \cap \mathcal{D}_{\iota'}(\mathbb{F}_p)$ for some $\iota, \iota' \in \mathcal{I}_2$, then the local integral on $\rho^{-1}(\tilde{x})$ is:

$$\begin{aligned}&\left(1-\frac{1}{p}\right)^{-1} \int_{\mathfrak{m}_p^2} |y|_p^{s_\iota - d_\iota} |z|_p^{s_{\iota'} - d_{\iota'}} \overline{\psi}_p\left(\frac{\alpha u}{y^{e_\iota} z^{e_{\iota'}}}\right) \lambda_S(\alpha y^{-1} z^{-1}) \mathbf{1}_{\mathbb{Z}_p}(\alpha y^{-1} z^{-1})\, \mathrm{d}y_p \mathrm{d}z_p \\ &\quad = \left(1-\frac{1}{p}\right) \int_{\mathfrak{m}_p^2} |y|_p^{s_\iota - d_\iota} |z|_p^{s_{\iota'} - d_{\iota'}} \lambda_S(\alpha y^{-1} z^{-1}) \int_{\mathbb{Z}_p^\times} \overline{\psi}_p\left(\frac{\alpha u b^{e_\iota}}{y^{e_\iota} z^{e_{\iota'}}}\right) \mathrm{d}b_p^\times \mathrm{d}y_p^\times \mathrm{d}z_p^\times.\end{aligned}$$

We can assume that $e_\iota \leq e_{\iota'}$. Then we can conclude that

$$\begin{aligned}\left|\int_{\rho^{-1}(\tilde{x})}\right| &\leq \int_{p^{-k} \leq |y^{e_\iota} z^{e_{\iota'}}|_p} |y^{e_\iota} z^{e_{\iota'}}|_p^{-\frac{1}{e_\iota}(1+\epsilon)}\, \mathrm{d}y_p^\times \mathrm{d}z_p^\times \\ &\quad + \int_{p^{-(k+1)} = |y^{e_\iota} z^{e_{\iota'}}|_p} |y^{e_\iota} z^{e_{\iota'}}|_p^{-\frac{1}{e_\iota}(1+\epsilon)} \left| \int_{\mathbb{Z}_p^\times} \overline{\psi}_p\left(\frac{\alpha u b^{e_\iota}}{y^{e_\iota} z^{e_{\iota'}}}\right) \mathrm{d}b^\times \right| \mathrm{d}y_p^\times \mathrm{d}z_p^\times \\ &\leq k^2 p^{\frac{k}{e_\iota}(1+\epsilon)} + k p^{\frac{k+1}{e_\iota}(1+\epsilon)} \times \begin{cases} 1 & \text{if } e_\iota > 2 \\ \frac{1}{\sqrt{p}-1} & \text{if } e_\iota = 2 \end{cases} \\ &\ll k^2 p^{\frac{2}{3}k(1+\epsilon)}.\end{aligned}$$

Thus our assertion follows from these estimates and Lemma 9.4 in [CLT02].

Next assume that p is a place of bad reduction or that p divides e_ι, for some $\iota \in \mathcal{I}_2$. Fix a compact subset of Ω_ϵ and assume that $\Re(\mathbf{s})$ is in that compact set. Choose a finite open covering $\{U_\eta\}$ of $\cup_{\iota \in \mathcal{I}_2} D_\iota(\mathbb{Q}_p)$ with analytic coordinates y_η, z_η such that the union of the boundary $D(\mathbb{Q}_p)$ and $E(\mathbb{Q}_p)$ is defined by $y_\eta = 0$ or $y_\eta \cdot z_\eta = 0$. Let V be the complement of $\cup_{\iota \in \mathcal{I}_2} D_\iota(\mathbb{Q}_p)$, and consider a partition of unity $\{\theta_\eta, \theta_V\}$ for $\{U_\eta, V\}$. Then it is clear that

$$\int_V \mathsf{H}_p(\mathbf{s}, g_p)^{-1} \overline{\psi}_p(\alpha x_p) \lambda_{S,p}(\alpha a_p) \mathbf{1}_{\frac{1}{N}\mathbb{Z}_p}(\alpha a_p) \theta_V \mathrm{d}g_p,$$

is bounded, so we need to study

$$\int_{U_\eta} \mathsf{H}_p(\mathbf{s}, g_p)^{-1}\overline{\psi}_p(\alpha x_p)\lambda_{S,p}(\alpha a_p)\mathbf{1}_{\frac{1}{N}\mathbb{Z}_p}(\alpha a_p)\theta_{U_\eta}\mathrm{d}g_p.$$

Assume that U_η meets only one $D_\iota(\mathbb{Q}_p)$ for some $\iota \in \mathcal{I}_2$. Then, the above integral looks like

$$\int_{U_\eta} = \int_{\mathbb{Q}_p^2} |y_\eta|_p^{s_\iota - d_\iota}\overline{\psi}_p(\alpha f/y_\eta^{e_\iota}))\lambda_{S,p}(\alpha g/y_\eta)\mathbf{1}_{\frac{1}{N}\mathbb{Z}_p}(\alpha g/y_\eta)\Phi(\mathbf{s}, y_\eta, z_\eta)\mathrm{d}y_{\eta,p}\mathrm{d}z_{\eta,p},$$

where f and g are nonvanishing analytic functions, and Φ is a smooth function with compact support. By shrinking U_η and changing variables, if necessary, we can assume that f and g are constant. The proof of Proposition 3.8 implies our assertion for this integral. Other integrals are treated similarly. □

Lemma 5.9. *For any compact set in an open convex cone Ω', defined by*

$$\begin{cases} s_\iota - d_\iota - 1 > 0 & \text{if } \iota \in \mathcal{I}_1 \\ s_\iota - d_\iota + 3 > 0 & \text{if } \iota \in \mathcal{I}_2 \\ s_\iota - d_\iota + 1 > 0 & \text{if } \iota \in \mathcal{I}_3 \end{cases}$$

there exists a constant $C > 0$ such that

$$|\widehat{\mathsf{H}}_\infty(\mathbf{s}, t, \alpha)| < \frac{C}{|\alpha|^2(1+t^2)},$$

for $\Re(\mathbf{s})$ in that compact set.

Proof. Consider the left invariant differential operators $\partial_a = a\partial/\partial a$ and $\partial_x = a\partial/\partial x$. Assume that $\Re(\mathbf{s}) \gg 0$. Integrating by parts, we have

$$\begin{aligned}\widehat{\mathsf{H}}_\infty(\mathbf{s}, t, \alpha) &= -\frac{1}{t^2}\int_{G(\mathbb{R})} \partial_a^2 \mathsf{H}_\infty(\mathbf{s}, g_\infty)^{-1}\overline{\psi}_\infty(\alpha x_\infty)|a_\infty|_\infty^{-it}\,\mathrm{d}g_\infty \\ &= \frac{1}{(2\pi)^2|\alpha|^2 t^2}\int_{G(\mathbb{R})} \frac{\partial^2}{\partial x^2}(\partial_a^2\mathsf{H}_\infty(\mathbf{s}, g_\infty)^{-1})\overline{\psi}_\infty(\alpha x_\infty)|a_\infty|_\infty^{-it}\,\mathrm{d}g_\infty.\end{aligned}$$

According to Proposition 2.2. in [CLT02],

$$\begin{aligned}\frac{\partial^2}{\partial x^2}(\partial_a^2\mathsf{H}_\infty(\mathbf{s}, g_\infty)^{-1}) &= |a|^{-2}\partial_x^2\partial_a^2\mathsf{H}_\infty(\mathbf{s}, g_\infty)^{-1} \\ &= \mathsf{H}_\infty(\mathbf{s} - 2\mathbf{m}(a), g_\infty)^{-1} \times (\text{a bounded smooth function}).\end{aligned}$$

Moreover, Lemma 4.4.1. of [CLT10] tells us that

$$\int_{G(\mathbb{R})} \mathsf{H}_\infty(\mathbf{s} - 2\mathbf{m}(a), g_\infty)^{-1}\mathrm{d}g_\infty,$$

is holomorphic on $\mathsf{T}_{\Omega'}$. Thus we can conclude our lemma. □

Lemma 5.10. *The Euler product*

$$\prod_p \widehat{\mathsf{H}}_p(\mathbf{s}, \lambda, t, \alpha) \cdot \widehat{\mathsf{H}}_\infty(\mathbf{s}, t, \alpha)$$

is holomorphic on $\mathsf{T}_{\Omega'}$.

Proof. First we prove that the Euler product is holomorphic on $\mathsf{T}_{\Omega'}$. To conclude this, we only need to discuss:

$$\prod_{p\notin S\cup S_3,\, |\alpha|_p=1,} \widehat{\mathsf{H}}_p(\mathbf{s},\lambda,t,\alpha),$$

where $S_3 = \{p : p \mid e_\iota \text{ for some } \iota \in \mathcal{I}_3\}$. Let p be a prime such that $p \notin S \cup S_3$ and $|\alpha|_p = 1$. Fix a compact subset of Ω', and assume that $\Re(s)$ is sitting in that compact set. From the definition of Ω', there exists $\epsilon > 0$ such that

$$\begin{cases} s_\iota - d_\iota + 1 > 2+\epsilon & \text{for any } \iota \in \mathcal{I}_1 \\ s_\iota - d_\iota + 1 > \epsilon & \text{for any } \iota \in \mathcal{I}_3. \end{cases}$$

Since we have

$$\{|a|_p \leq 1\} = X(\mathbb{Q}_p) \setminus \rho^{-1}(\cup_{\iota\in\mathcal{I}_2}\mathcal{D}_\iota(\mathbb{F}_p)),$$

we can conclude that

$$\widehat{\mathsf{H}}_p(\mathbf{s},\lambda,\alpha) = \sum_{\tilde{x}\notin\cup_{\iota\in\mathcal{I}_2}\mathcal{D}_\iota(\mathbb{F}_p)} \int_{\rho^{-1}(\tilde{x})} \mathsf{H}_p(\mathbf{s},g_p)^{-1}\overline{\psi}_p(\alpha x_p)\lambda_S(a_p)\mathrm{d}g_p.$$

It is easy to see that

$$\sum_{\tilde{x}\notin\cup_{\iota\in\mathcal{I}}\mathcal{D}_\iota(\mathbb{F}_p)} \int_{\rho^{-1}(\tilde{x})} = \int_{G(\mathbb{Z}_p)} 1\,\mathrm{d}g_p = 1.$$

Also it follows from a formula of J. Denef (see [Den87, Theorem 3.1] or [CLT10, Proposition 4.1.7]) and Lemma 9.4 in [CLT02] that there exists an uniform bound $C > 0$ such that for any $\tilde{x} \in \cup_{\iota\in\mathcal{I}_1}\mathcal{D}_\iota(\mathbb{F}_p)$,

$$\left|\int_{\rho^{-1}(\tilde{x})}\right| < \int_{\rho^{-1}(\tilde{x})} \mathsf{H}_p(\Re(\mathbf{s}),g_p)^{-1}\mathrm{d}g_p < \frac{C}{p^{2+\epsilon}}.$$

Hence we need to obtain uniform bounds of $\int_{\rho^{-1}(\tilde{x})}$ for

$$\tilde{x} \in \cup_{\iota\in\mathcal{I}_3}\mathcal{D}_\iota(\mathbb{F}_p) \setminus \cup_{\iota\in\mathcal{I}_1\cup\mathcal{I}_2}\mathcal{D}_\iota(\mathbb{F}_p).$$

Let $\tilde{x} \in \mathcal{D}_\iota(\mathbb{F}_p)$ for some $\iota \in \mathcal{I}_3$, but $\tilde{x} \notin \cup_{\iota\in\mathcal{I}_1\cup\mathcal{I}_2}\mathcal{D}_\iota(\mathbb{F}_p) \cup \mathcal{E}(\mathbb{F}_p)$. Then it follows from Lemmas 4.2 and 5.7 that

$$\begin{aligned} \int_{\rho^{-1}(\tilde{x})} &= \left(1-\frac{1}{p}\right)^{-1} \int_{\mathfrak{m}_p^2} |y|_p^{s_\iota-d_\iota}\overline{\psi}_p(u/y^{e_\iota})\,\mathrm{d}y_p\mathrm{d}z_p \\ &= \frac{1}{p-1}\int_{\mathfrak{m}_p} |y|_p^{s_\iota-d_\iota}\int_{\mathbb{Z}_p^\times} \overline{\psi}_p(ub^{e_\iota}/y^{e_\iota})\,\mathrm{d}b_p^\times\mathrm{d}y_p \\ &= \begin{cases} 0 & \text{if } e_\iota > 1 \\ -\frac{p^{-(s_\iota-d_\iota+2)}}{p-1} & \text{if } e_\iota = 1. \end{cases} \end{aligned}$$

If $\tilde{x} \in \mathcal{D}_\iota(\mathbb{F}_p) \cap \mathcal{E}(\mathbb{F}_p)$ for some $\iota \in \mathcal{I}_3$, then we have

$$\begin{aligned}\int_{\rho^{-1}(\tilde{x})} &= \left(1 - \frac{1}{p}\right)^{-1} \int_{\mathfrak{m}_p^2} |y|_p^{s_\iota - d_\iota} \overline{\psi}_p(z/y^{e_\iota}) \mathrm{d}y_p \mathrm{d}z_p \\ &= \left(1 - \frac{1}{p}\right)^{-1} \int_{m_p} |y|_p^{s_\iota - d_\iota} \int_{\mathfrak{m}_p} \overline{\psi}_p(z/y^{e_\iota})\, \mathrm{d}z_p \mathrm{d}y_p \\ &= \begin{cases} 0 & \text{if } e_\iota > 1 \\ p^{-(s_\iota - d_\iota + 2)} & \text{if } e_\iota = 1. \end{cases}\end{aligned}$$

If $\tilde{x} \in \mathcal{D}_\iota(\mathbb{F}_p) \cap \mathcal{D}_{\iota'}(\mathbb{F}_p)$ for some $\iota, \iota' \in \mathcal{I}_3$, then it follows from Lemma 5.7 that

$$\begin{aligned}\int_{\rho^{-1}(\tilde{x})} &= \left(1 - \frac{1}{p}\right)^{-1} \int_{\mathfrak{m}_p^2} |y|_p^{s_\iota - d_\iota} |z|_p^{s_{\iota'} - d_{\iota'}} \overline{\psi}_p \left(\frac{u}{y^{e_\iota} z^{e_{\iota'}}} \right) \mathrm{d}y_p \mathrm{d}z_p \\ &= \left(1 - \frac{1}{p}\right)^{-1} \int_{\mathfrak{m}_p^2} |y|_p^{s_\iota - d_\iota} |z|_p^{s_{\iota'} - d_{\iota'}} \int_{\mathbb{Z}_p^\times} \overline{\psi}_p \left(\frac{u b^{e_\iota}}{y^{e_\iota} z^{e_{\iota'}}} \right) \mathrm{d}b_p^\times\, \mathrm{d}y_p \mathrm{d}z_p \\ &= 0.\end{aligned}$$

Thus we can conclude from these estimates and Lemma 9.4 in [CLT02] that there exists an uniform bound $C' > 0$ such that

$$\left| \widehat{\mathsf{H}}_p(\mathbf{s}, \lambda, t, \alpha) - 1 \right| < \frac{C'}{p^{1+\epsilon}}$$

Our assertion follows from this. $\square$

Lemma 5.11. *Let Ω'_ϵ be an open convex cone, defined by*

$$\begin{cases} s_\iota - d_\iota - 2 - \epsilon > 0 & \text{if } \iota \in \mathcal{I}_1 \\ s_\iota - d_\iota + 2 + 2\epsilon > 0 & \text{if } \iota \in \mathcal{I}_2 \\ s_\iota - d_\iota + 1 > 0 & \text{if } \iota \in \mathcal{I}_3 \end{cases}$$

where $\epsilon > 0$ is sufficiently small. Fix a compact subset of Ω'_ϵ and $\epsilon \gg \delta > 0$. Then there exists a constant $C > 0$ such that

$$\left| \prod_p \widehat{\mathsf{H}}_p(\mathbf{s}, \lambda, t, \alpha) \cdot \widehat{\mathsf{H}}_\infty(\mathbf{s}, \alpha, t) \right| < \frac{C}{(1+t^2)|\beta|^{\frac{4}{3} - \frac{8}{3}\epsilon - \delta} |\gamma|^{1+\epsilon-\delta}},$$

for $\Re(s)$ in that compact set, where $\alpha = \frac{\beta}{\gamma}$ with $\gcd(\beta, \gamma) = 1$.

Proof. This lemma follows from Lemmas 5.6, 5.8, and 5.9, and from the proof of Lemma 5.10. $\square$

Theorem 5.12. *The zeta function $\mathsf{Z}_1(\mathbf{s}, \mathrm{id})$ is holomorphic on the tube domain over an open neighborhood of the shifted effective cone $-K_X + \Lambda_{\mathrm{eff}}(X)$.*

Proof. Let $1 \gg \epsilon \gg \delta > 0$. Lemma 5.11 implies that

$$\mathsf{Z}_1(\mathbf{s}, \mathrm{id}) = \sum_{\lambda \in \mathbf{M},\, \lambda(-1) = 1} \sum_{\alpha \in \mathbb{Q}^\times} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \prod_p \widehat{\mathsf{H}}_p(\mathbf{s}, \lambda, t, \alpha) \cdot \widehat{\mathsf{H}}_\infty(\mathbf{s}, t, \alpha)\, \mathrm{d}t,$$

is absolutely and uniformly convergent on Ω'_ϵ, so $\mathsf{Z}_1(\mathbf{s}, \mathrm{id})$ is holomorphic on $\mathsf{T}_{\Omega'_\epsilon}$. Now note that the image of Ω'_ϵ by $\mathrm{Pic}^G(X) \to \mathrm{Pic}(X)$ contains an open neighborhood of $-K_X + \Lambda_{\mathrm{eff}}(X)$. Thus our theorem is concluded. $\square$

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